CHAPTER 11
Limits and an Introduction to Calculus

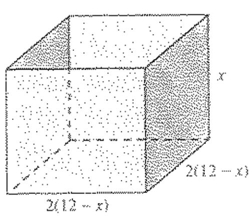
Section 11.1 Introduction to Limits

- If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L:
  $$\lim_{x \to c} f(x) = L.$$
- You should be able to use a calculator to find a limit.
- You should be able to use a graph to find a limit.
- You should understand how limits can fail to exist:
  (a) $f(x)$ approaches a different number from the right of c than it approaches from the left of c.
  (b) $f(x)$ increases or decreases without bound as x approaches c.
  (c) $f(x)$ oscillates between two fixed values as x approaches c.
- You should know and be able to use the elementary properties of limits.

Vocabulary Check

1. limit
2. oscillates
3. direct substitution

1. (a)

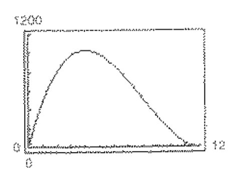


(b) $V = (\text{base})\text{height} = (24 - 2x)^2 x = 4x(12 - x)^2$

(c) $\lim_{x \to 4} V = 1024$

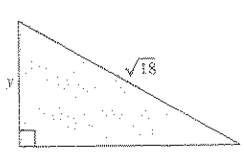
x	3	3.5	3.9	4	4.1	4.5	5
V	972.0	1011.5	1023.5	1024.0	1023.5	1012.5	980.0

(d)



Maximum at $x = 4$

2. (a)



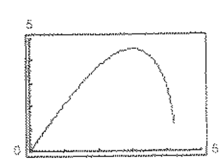
(b) $x^2 + y^2 = 18 \implies y = \sqrt{18 - x^2}$

Area $= \frac{1}{2}bh = \frac{1}{2}x\sqrt{18 - x^2}$

(c) $\lim_{x \to 3} A(x) = 4.5$

x	2	2.5	2.9	3	3.1	3.5	4
A	3.74	4.28	4.49	4.5	4.49	4.20	2.83

(d)



3. $\lim_{x \to 2} (5x + 4) = 14$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	13.5	13.95	13.995	14	14.005	14.05	14.5

The limit is reached.

4. $\lim_{x \to 1} (2x^2 + x - 4) = -1$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	-1.48	-1.0498	-1.0050	-1	-0.9950	-0.9498	-0.48

The limit is reached.

5. $\lim_{x \to 3} \dfrac{x - 3}{x^2 - 9} = \dfrac{1}{6}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	0.1695	0.1669	0.16669	Error	0.16664	0.1664	0.1639

The limit is not reached.

6. $\lim_{x \to -1} \dfrac{x + 1}{x^2 - x - 2} = -\dfrac{1}{3}$

x	-1.1	-1.01	-1.001	-1.0	-0.999	-0.99	-0.9
$f(x)$	-0.3226	-0.3322	-0.3332	Error	-0.3334	-0.3344	-0.3348

The limit is not reached.

7. $\lim_{x \to 0} \dfrac{\sin 2x}{x} = 2$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	1.9867	1.99987	1.9999987	Error	1.9999987	1.99987	1.987

The limit is not reached.

8. $\lim_{x \to 0} \dfrac{\tan x}{2x} = \dfrac{1}{2}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5017	0.50002	0.5	Error	0.5	0.50002	0.5017

The limit is not reached.

9. $\lim_{x \to 0} \frac{e^{2x} - 1}{x} = 2$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	1.8127	1.9801	1.9980	Error	2.0020	2.0201	2.2140

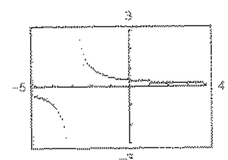
The limit is not reached.

10. $\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.0536	1.0050	1.0005	Error	0.9995	0.9950	0.9531

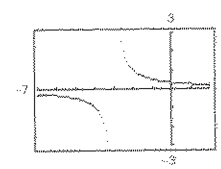
The limit is not reached.

11. $\lim_{x \to 1} \frac{x - 1}{x^2 + 2x - 3} = \frac{1}{4}$



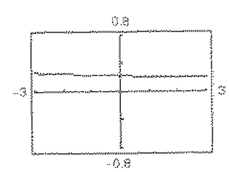
x	0.9	0.99	0.999	1.0	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	Error	0.2499	0.2494	0.2439

12. $\lim_{x \to -2} \frac{x + 2}{x^2 + 5x + 6} = 1$



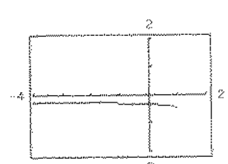
x	-2.1	-2.01	-2.001	-2.0	-1.999	-1.99	-1.9
$f(x)$	1.1111	1.0101	1.0010	Error	0.9990	0.9901	0.9091

13. $\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} \approx 0.2236 \left(\text{Actual limit is } \frac{1}{2\sqrt{5}}.\right)$



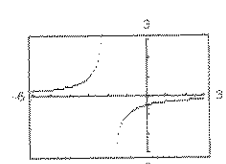
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.2247	0.2237	0.2236	Error	0.2236	0.2235	0.2225

14. $\lim_{x \to -3} \frac{\sqrt{1 - x} - 2}{x + 3} = -\frac{1}{4}$



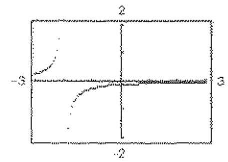
x	-3.1	-3.01	-3.001	-3.0	-2.999	-2.99	-2.9
$f(x)$	-0.2485	-0.2498	-0.25	Error	-0.25	-0.2502	-0.2516

15. $\lim_{x \to -4} \frac{[x/(x + 2)] - 2}{x + 4} = \frac{1}{2}$



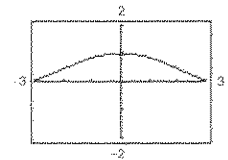
x	-4.1	-4.01	-4.001	-4.0	-3.999	-3.99	-3.9
$f(x)$	0.4762	0.4975	0.4998	Error	0.5003	0.5025	0.5263

16. $\lim\limits_{x \to 2} \dfrac{\dfrac{1}{x+2} - \dfrac{1}{4}}{x - 2} = -\dfrac{1}{16}$



x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	-0.0641	-0.0627	-0.0625	Error	-0.0625	-0.0623	-0.0610

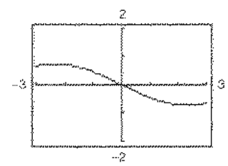
17. Make sure your calculator is set in radian mode.



$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	0.9999998	Error	0.9999998	0.99998	0.9983

18. $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.050	0.005	0.0005	Error	-0.0005	-0.005	-0.050

19. $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x} = 0$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.0997	-0.0100	-0.0010	Error	0.0010	0.0100	0.0997

20. $\lim\limits_{x \to 0} \dfrac{2x}{\tan 4x} = 0.5$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.4730	0.4997	0.499997	Error	0.499997	0.4997	0.4730

21. $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{2x} = 1.0$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.9063	0.9901	0.9990	Error	1.0010	1.0101	1.1070

22. $\lim\limits_{x \to 0} \dfrac{1 - e^{-4x}}{x} = 4.0$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	4.9182	4.0811	4.0080	Error	3.9920	3.9211	3.2968

23. $\lim\limits_{x \to 1} \dfrac{\ln(2x-1)}{x-1} = 2$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	2.2314	2.0203	2.0020	Error	1.9980	1.9803	1.8232

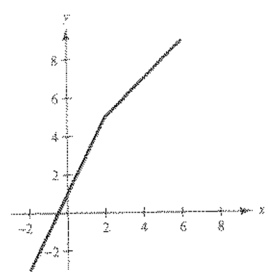
24. $\lim\limits_{x \to 1} \dfrac{\ln x^2}{x-1} = 2.0$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	2.1072	2.0101	2.0010	Error	1.9990	1.9901	1.9062

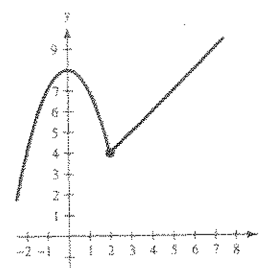
25. $f(x) = \begin{cases} 2x+1, & x < 2 \\ x+3, & x \geq 2 \end{cases}$

The limit exists as x approaches 2:

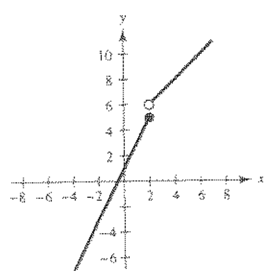
$\lim\limits_{x \to 2} f(x) = 5$



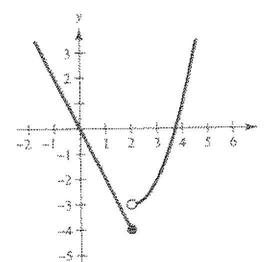
26. $\lim\limits_{x \to 2} f(x) = 4$



27. $\lim\limits_{x \to 2} f(x)$ does not exist.



28. $\lim\limits_{x \to 2} f(x)$ does not exist.



29. $\lim\limits_{x \to -4} (x^2 - 3) = 13$

30. $\lim\limits_{x \to 2} \dfrac{3x^2 - 12}{x-2} = 12$

31. $\lim\limits_{x \to -2} \dfrac{|x+2|}{x+2}$ does not exist. $f(x) = \dfrac{|x+2|}{x+2}$ equals -1 to the left of -2, and equals 1 to the right of -2.

32. The limit does not exist because $f(x)$ does not approach a real number as x approaches 1.

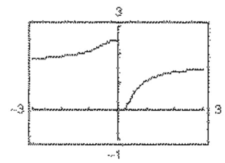
33. The limit does not exist because $f(x)$ oscillates between 2 and -2.

34. $\lim\limits_{x \to -1} \sin\left(\dfrac{\pi x}{2}\right) = -1$

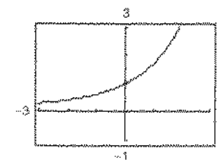
35. $\lim\limits_{x \to \pi/2} \tan x$ does not exist.

36. $\lim\limits_{x \to \pi/2} \sec x$ does not exist.

37. $\lim_{x \to 0} \frac{5}{2 + e^{1/x}}$ does not exist.

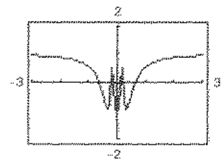


38. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

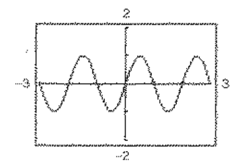


39. $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist.

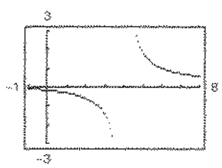
The graph oscillates between -1 and 1.



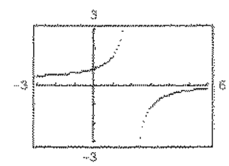
40. $\lim_{x \to -1} \sin \pi x = 0$



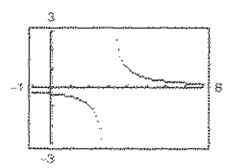
41. $\lim_{x \to 4} \frac{\sqrt{x + 3} - 1}{x - 4}$ does not exist.



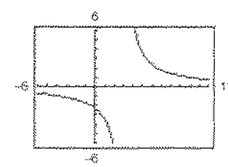
42. $\lim_{x \to 2} \frac{\sqrt{x + 5} - 4}{x - 2}$ does not exist.



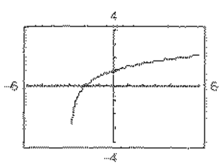
43. $\lim_{x \to 1} \frac{x - 1}{x^2 - 4x + 3} = -\frac{1}{2}$



44. $\lim_{x \to 3} \frac{7}{x - 3}$ does not exist.

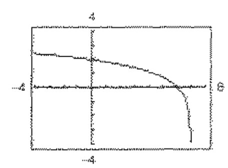


45. $\lim_{x \to 4} \ln(x + 3) \approx 1.946$ (Exact limit is $\ln 7$.)



46. $\lim_{x \to -1} \ln(7 - x) = \ln(7 - (-1))$

$= \ln 8$



47. (a) $\lim_{x \to c} [-2g(x)] = -2(6) = -12$

(b) $\lim_{x \to c} [f(x) + g(x)] = 3 + 6 = 9$

(c) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{3}{6} = \frac{1}{2}$

(d) $\lim_{x \to c} \sqrt{f(x)} = \sqrt{3}$

48. (a) $\lim_{x \to c} [f(x) + g(x)]^2 = (5 - 2)^2 = 9$

(b) $\lim_{x \to c} [6f(x)g(x)] = 6(5)(-2) = -60$

(c) $\lim_{x \to c} \frac{5g(x)}{4f(x)} = \frac{5(-2)}{4(5)} = -\frac{1}{2}$

(d) $\lim_{x \to c} \frac{1}{\sqrt{f(x)}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

49. (a) $\lim_{x \to 2} f(x) = 2^3 = 8$

(b) $\lim_{x \to 2} g(x) = \frac{\sqrt{2^2 + 5}}{2(2^2)} = \frac{3}{8}$

(c) $\lim_{x \to 2} [f(x)g(x)] = 8\left(\frac{3}{8}\right) = 3$

(d) $\lim_{x \to 2} [g(x) - f(x)] = \frac{3}{8} - 8 = -\frac{61}{8}$

50. (a) $\lim_{x \to 2} f(x) = \dfrac{2}{3 - 2} = 2$

(b) $\lim_{x \to 2} g(x) = \sin(\pi 2) = 0$

(c) $\lim_{x \to 2} [f(x)g(x)] = 2(0) = 0$

(d) $\lim_{x \to 2} [g(x) - f(x)] = 0 - 2 = -2$

51. $\lim_{x \to 5} (10 - x^2) = 10 - 5^2 = -15$

52. $\lim_{x \to -2} \left(\dfrac{1}{2}x^3 - 5x\right) = \dfrac{1}{2}(-2)^3 - 5(-2) = 6$

53. $\lim_{x \to -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 7$

54. $\lim_{x \to -2} (x^3 - 6x + 5) = (-2)^3 - 6(-2) + 5 = 9$

55. $\lim_{x \to 3} \left(-\dfrac{9}{x}\right) = -\dfrac{9}{3} = -3$

56. $\lim_{x \to -5} \dfrac{6}{x + 2} = \dfrac{6}{-5 + 2} = \dfrac{6}{-3} = -2$

57. $\lim_{x \to -3} \dfrac{3x}{x^2 + 1} = -\dfrac{9}{10}$

58. $\lim_{x \to 4} \dfrac{x - 1}{x^2 + 2x + 3} = \dfrac{4 - 1}{16 + 8 + 3} = \dfrac{3}{27} = \dfrac{1}{9}$

59. $\lim_{x \to -2} \dfrac{5x + 3}{2x - 9} = \dfrac{5(-2) + 3}{2(-2) - 9} = \dfrac{-7}{-13} = \dfrac{7}{13}$

60. $\lim_{x \to 3} \dfrac{x^2 + 1}{x} = \dfrac{9 + 1}{3} = \dfrac{10}{3}$

61. $\lim_{x \to -1} \sqrt{x + 2} = \sqrt{-1 + 2} = 1$

62. $\lim_{x \to 3} \sqrt[3]{x^2 - 1} = \sqrt[3]{9 - 1} = 2$

63. $\lim_{x \to 7} \dfrac{5x}{\sqrt{x + 2}} = \dfrac{5(7)}{\sqrt{7 + 2}} = \dfrac{35}{3}$

64. $\lim_{x \to 8} \dfrac{\sqrt{x + 1}}{x - 4} = \dfrac{\sqrt{8 + 1}}{x - 4} = \dfrac{3}{4}$

65. $\lim_{x \to 3} e^x = e^3 \approx 20.0855$

66. $\lim_{x \to e} \ln x = \ln e = 1$

67. $\lim_{x \to \pi} \sin 2x = \sin 2\pi = 0$

68. $\lim_{x \to \pi} \tan x = \tan \pi = 0$

69. $\lim_{x \to 1/2} \arcsin x = \arcsin \dfrac{1}{2} = \dfrac{\pi}{6} \approx 0.5236$

70. $\lim_{x \to 1} \arccos \dfrac{x}{2} = \arccos \dfrac{1}{2} = \dfrac{\pi}{3} \approx 1.0472$

71. True

72. True (assuming the limits exist).

73. Answers will vary.

74. In general, you cannot use a graphing utility to determine whether a limit can be reached. It is important to analyze a function analytically.

75. (a) No. The limit may or may not exist, and if it does exist, it may not equal 4.

(b) No. $f(2)$ may or may not exist, and if $f(2)$ exists, it may not equal 4.

76. $\lim_{x \to 5} f(x) = 12$ means that the values of f approach 12 as x approaches 5.

77. $\dfrac{5-x}{3x-15} = \dfrac{5-x}{-3(5-x)} = -\dfrac{1}{3},\ x \neq 5$

78. $\dfrac{x^2-81}{9-x} = \dfrac{(x-9)(x+9)}{9-x} = -x-9,\ x \neq 9$

79. $\dfrac{15x^2+7x-4}{15x^2+x-2} = \dfrac{(3x-1)(5x+4)}{(3x-1)(5x+2)}$

$= \dfrac{5x+4}{5x+2},\ x \neq \dfrac{1}{3}$

80. $\dfrac{x^2-12x+36}{x^2-7x+6} = \dfrac{(x-6)(x-6)}{(x-6)(x-1)} = \dfrac{x-6}{x-1},\ x \neq 6$

81. $\dfrac{x^2+27}{x^2+x-6} = \dfrac{(x+3)(x^2-3x+9)}{(x+3)(x-2)}$

$= \dfrac{x^2-3x+9}{x-2},\ x \neq -3$

82. $\dfrac{x^3-8}{x^2-4} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$

$= \dfrac{x^2+2x+4}{x+2},\ x \neq 2$

Section 11.2 Techniques for Evaluating Limits

- You can use direct substitution to find the limit of a polynomial function $p(x)$:
  $\lim\limits_{x\to c} p(x) = p(c)$.
- You can use direct substitution to find the limit of a rational function $r(x) = \dfrac{p(x)}{q(x)}$, as long as $q(c) \neq 0$:
  $\lim\limits_{x\to c} r(x) = r(c) = \dfrac{p(c)}{q(c)},\ q(c) \neq 0$.
- You should be able to use cancellation techniques to find a limit.
- You should know how to use rationalization techniques to find a limit.
- You should know how to use technology to find a limit.
- You should be able to calculate one-sided limits.

Vocabulary Check

1. dividing out technique
2. indeterminate form
3. one-sided limit
4. difference quotient

1. $g(x) = \dfrac{-2x^2+x}{x},\ g_2(x) = -2x+1$

(a) $\lim\limits_{x\to 0} g(x) = 1$

(b) $\lim\limits_{x\to -1} g(x) = 3$

(c) $\lim\limits_{x\to -2} g(x) = 5$

2. $h(x) = \dfrac{x^2-3x}{x},\ h_2(x) = x-3$

(a) $\lim\limits_{x\to -2} h(x) = -5$

(b) $\lim\limits_{x\to 0} h(x) = -3$

(c) $\lim\limits_{x\to 3} h(x) = 0$

3. $g(x) = \dfrac{x^3 - x}{x - 1}$, $g_2(x) = x^2 + x = x(x + 1)$

(a) $\lim\limits_{x \to 1} g(x) = 2$

(b) $\lim\limits_{x \to -1} g(x) = 0$

(c) $\lim\limits_{x \to 0} g(x) = 0$

4. $f(x) = \dfrac{x^2 - 1}{x + 1}$, $f_2(x) = x - 1$

(a) $\lim\limits_{x \to 1} f(x) = 0$

(b) $\lim\limits_{x \to 2} f(x) = 1$

(c) $\lim\limits_{x \to -1} f(x) = -2$

5. $$\lim_{x \to 6} \frac{x - 6}{x^2 - 36} = \lim_{x \to 6} \frac{x - 6}{(x - 6)(x + 6)} = \lim_{x \to 6} \frac{1}{x + 6} = \frac{1}{12}$$

6. $$\lim_{x \to 9} \frac{9 - x}{x^2 - 81} = \lim_{x \to 9} \frac{9 - x}{(x - 9)(x + 9)} = \lim_{x \to 9} \frac{-1}{x + 9} = \frac{-1}{18}$$

7. $$\lim_{x \to -1} \frac{1 - 2x - 3x^2}{1 + x} = \lim_{x \to -1} \frac{(1 + x)(1 - 3x)}{1 + x} = \lim_{x \to -1} (1 - 3x) = 4$$

8. $$\lim_{x \to -4} \frac{2x^2 + 7x - 4}{x + 4} = \lim_{x \to -4} \frac{(x + 4)(2x - 1)}{x + 4} = \lim_{x \to -4} (2x - 1) = -9$$

9. $$\lim_{t \to 2} \frac{t^3 - 8}{t - 2} = \lim_{t \to 2} \frac{(t - 2)(t^2 + 2t + 4)}{t - 2} = \lim_{t \to 2} (t^2 + 2t + 4) = 4 + 4 + 4 = 12$$

10. $$\lim_{a \to -4} \frac{a^3 + 64}{a + 4} = \lim_{a \to -4} \frac{(a + 4)(a^2 - 4a + 16)}{a + 4} = \lim_{a \to -4} (a^2 - 4a + 16) = 16 + 16 + 16 = 48$$

11. $$\lim_{x \to 1} \frac{x^4 - 1}{x^4 - 3x^2 - 4} = \frac{0}{-6} = 0$$

12. $$\lim_{x \to 2} \frac{x^4 - 2x^2 - 8}{x^4 - 6x^2 + 8} = \lim_{x \to 2} \frac{(x - 2)(x + 2)(x^2 + 2)}{(x - 2)(x + 2)(x^2 - 2)} = \lim_{x \to 2} \frac{(x + 2)(x^2 + 2)}{(x + 2)(x^2 - 2)} = \frac{4(6)}{4(2)} = 3$$

13. $$\lim_{x \to -1} \frac{x^3 + 2x^2 - x - 2}{x^3 + 4x^2 - x - 4} = \lim_{x \to -1} \frac{(x - 1)(x + 1)(x + 2)}{(x - 1)(x + 1)(x + 4)} = \lim_{x \to -1} \frac{(x - 1)(x + 2)}{(x - 1)(x + 4)} = \frac{(-2)(1)}{(-2)(3)} = \frac{1}{3}$$

14. $$\lim_{x \to -3} \frac{x^3 + 2x^2 - 9x - 18}{x^3 + x^2 - 9x - 9} = \lim_{x \to -3} \frac{(x - 3)(x + 2)(x + 3)}{(x - 3)(x + 1)(x + 3)} = \lim_{x \to -3} \frac{x + 2}{x + 1} = \frac{-1}{-2} = \frac{1}{2}$$

15. $\lim_{x\to 2}\dfrac{x^3+2x^2-5x-6}{x^3-7x+6}=\lim_{x\to 2}\dfrac{(x-2)(x+1)(x+3)}{(x-2)(x-1)(x+3)}$

$=\lim_{x\to 2}\dfrac{x+1}{x-1}$

$=\dfrac{3}{1}=3$

16. $\lim_{x\to 3}\dfrac{x^3-4x^2-3x+18}{x^3-4x^2+x+6}=\lim_{x\to 3}\dfrac{(x-3)^2(x+2)}{(x-3)(x-2)(x+1)}$

$=\lim_{x\to 3}\dfrac{(x-3)(x+2)}{(x-2)(x+1)}$

$=\dfrac{0}{4}=0$

17. $\lim_{y\to 0}\dfrac{\sqrt{5+y}-\sqrt{5}}{y}=\lim_{y\to 0}\dfrac{\sqrt{5+y}-\sqrt{5}}{y}\cdot\dfrac{\sqrt{5+y}+\sqrt{5}}{\sqrt{5+y}+\sqrt{5}}$

$=\lim_{y\to 0}\dfrac{(5+y)-5}{y\left(\sqrt{5+y}+\sqrt{5}\right)}$

$=\lim_{y\to 0}\dfrac{1}{\sqrt{5+y}+\sqrt{5}}$

$=\dfrac{1}{2\sqrt{5}}=\dfrac{\sqrt{5}}{10}$

18. $\lim_{z\to 0}\dfrac{\sqrt{7-z}-\sqrt{7}}{z}\left(\dfrac{\sqrt{7-z}+\sqrt{7}}{\sqrt{7-z}+\sqrt{7}}\right)=\lim_{z\to 0}\dfrac{(7-z)-7}{z\left(\sqrt{7-z}+\sqrt{7}\right)}$

$=\lim_{z\to 0}\dfrac{-1}{\sqrt{7-z}+\sqrt{7}}$

$=\dfrac{-1}{2\sqrt{7}}=-\dfrac{\sqrt{7}}{14}$

19. $\lim_{x\to -3}\dfrac{\sqrt{x+7}-2}{x+3}=\lim_{x\to -3}\dfrac{\sqrt{x+7}-2}{x+3}\cdot\dfrac{\sqrt{x+7}+2}{\sqrt{x+7}+2}$

$=\lim_{x\to -3}\dfrac{(x+7)-4}{(x+3)\left(\sqrt{x+7}+2\right)}$

$=\lim_{x\to -3}\dfrac{1}{\sqrt{x+7}+2}=\dfrac{1}{4}$

20. $\lim_{x\to 2}\dfrac{4-\sqrt{18-x}}{x-2}=\lim_{x\to 2}\dfrac{4-\sqrt{18-x}}{x-2}\cdot\dfrac{4+\sqrt{18-x}}{4+\sqrt{18-x}}$

$=\lim_{x\to 2}\dfrac{16-(18-x)}{(x-2)\left(4+\sqrt{18-x}\right)}$

$=\lim_{x\to 2}\dfrac{1}{4+\sqrt{18-x}}=\dfrac{1}{8}$

21. $\displaystyle \lim_{x\to 0} \frac{1/(1+x)-1}{x} = \lim_{x\to 0} \frac{1-(1+x)}{(1+x)x}$

$\displaystyle = \lim_{x\to 0} \frac{-1}{1+x} = -1$

22. $\displaystyle \lim_{x\to 0} \frac{\frac{1}{x-8}+\frac{1}{8}}{x} = \lim_{x\to 0} \frac{8+(x-8)}{(x-8)(8)x}$

$\displaystyle = \lim_{x\to 0} \frac{1}{(x-8)8} = -\frac{1}{64}$

23. $\displaystyle \lim_{x\to 0} \frac{\sec x}{\tan x} = \lim_{x\to 0} \frac{1}{\cos x} \cdot \frac{\cos x}{\sin x}$

$\displaystyle = \lim_{x\to 0} \frac{1}{\sin x}$, does not exist

24. $\displaystyle \lim_{x\to \pi/2} \frac{1-\sin x}{\cos x} = \lim_{x\to \pi/2} \frac{1-\sin x}{\cos x} \cdot \frac{1+\sin x}{1+\sin x}$

$\displaystyle = \lim_{x\to \pi/2} \frac{1-\sin^2 x}{\cos x(1+\sin x)}$

$\displaystyle = \lim_{x\to \pi/2} \frac{\cos^2 x}{\cos x(1+\sin x)}$

$\displaystyle = \lim_{x\to \pi/2} \frac{\cos x}{1+\sin x} = 0$

25. $\displaystyle \lim_{x\to 0} \frac{\cos 2x}{\cot 2x} = \lim_{x\to 0} \frac{\cos 2x}{(\cos 2x)/\sin(2x)}$

$\displaystyle = \lim_{x\to 0} \sin 2x = 0$

26. $\displaystyle \lim_{x\to 0} \frac{\sin x - x}{\sin x} = \lim_{x\to 0} \left(1 - \frac{x}{\sin x}\right)$

$= 1 - 1 = 0$

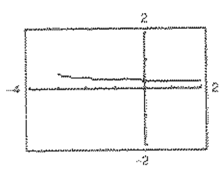
27. $\displaystyle \lim_{x\to \pi/2} \frac{\sin x - 1}{x} = \frac{1-1}{\pi/2} = 0$

28. $\displaystyle \lim_{x\to \pi} \frac{1+\cos x}{x} = \frac{1+(-1)}{\pi} = 0$

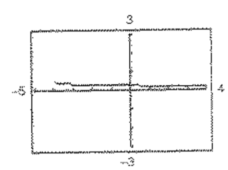
29. $\displaystyle f(x) = \frac{\sqrt{x+3}-\sqrt{3}}{x}$

$\displaystyle \lim_{x\to 0} f(x) \approx 0.2887$

$\displaystyle \left(\text{Exact limit: } \frac{1}{2\sqrt{3}}\right)$

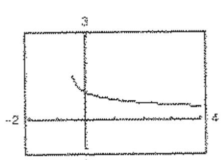


30. $\displaystyle \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} \approx 0.25, \quad \left(\frac{1}{4}\right)$

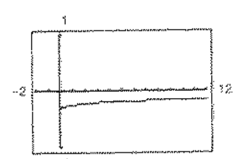


31. $\displaystyle f(x) = \frac{\sqrt{2x+1}-1}{x}$

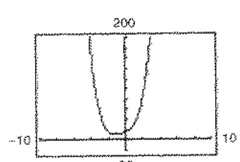
$\displaystyle \lim_{x\to 0} f(x) = 1$



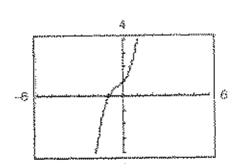
32. $\displaystyle \lim_{x\to 9} \frac{3-\sqrt{x}}{x-9} = -0.1667, \quad \left(-\frac{1}{6}\right)$



33. $\displaystyle \lim_{x\to 2} \frac{x^5-32}{x-2} = 80$

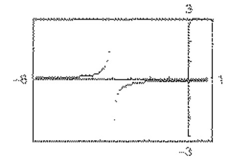


34. $\displaystyle \lim_{x\to 1} \frac{x^4-1}{x-1} = 4$

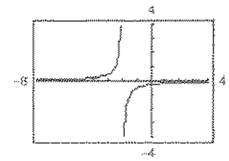


35. $f(x) = \dfrac{1/(x+4) - (1/4)}{x}$

$\lim\limits_{x \to 0} f(x) = -\dfrac{1}{16}, (-0.0625)$

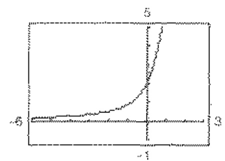


36. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x} = -0.25, \left(-\dfrac{1}{4}\right)$

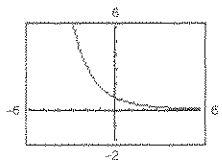


37. $f(x) = \dfrac{e^{2x} - 1}{x}$

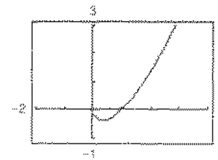
$\lim\limits_{x \to 0} f(x) = 2$



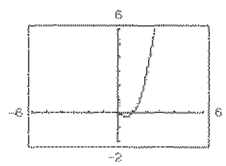
38. $\lim\limits_{x \to 0} \dfrac{1 - e^{-x}}{x} = 1$



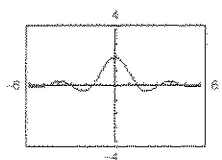
39. $\lim\limits_{x \to 0^+} x \ln x = 0$



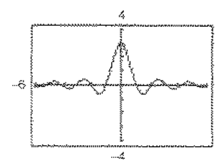
40. $\lim\limits_{x \to 0^+} x^2 \ln x = 0$



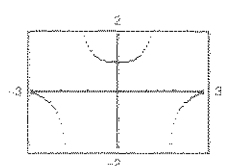
41. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = 2$



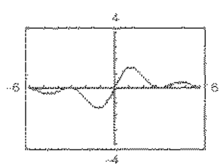
42. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x} = 3$



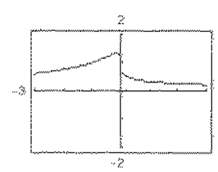
43. $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$



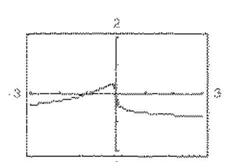
44. $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x} = 0$



45. $\lim\limits_{x \to 1} \dfrac{1 - \sqrt[3]{x}}{1 - x} = \dfrac{1}{3} \approx 0.333$

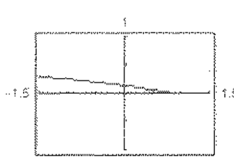


46. $\lim\limits_{x \to 1} \dfrac{\sqrt[3]{x} - x}{x - 1} \approx -0.667, \left(-\dfrac{2}{3}\right)$

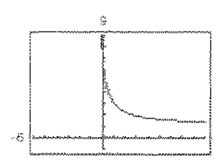


47. $f(x) = (1 - x)^{2/x}$

$\lim\limits_{x \to 0} f(x) \approx 0.135$

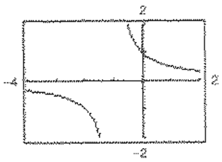


48. $\lim\limits_{x \to 0} (1 + 2x)^{1/x} \approx 7.389$



49. $f(x) = \dfrac{x-1}{x^2-1}$

(a) Graphically, $\lim\limits_{x \to 1^-} \dfrac{x-1}{x^2-1} = \dfrac{1}{2}$.



(b)

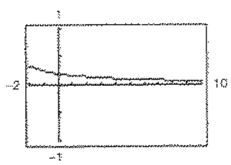
x	0.5	0.9	0.99	0.999	1
$f(x)$	0.6667	0.5263	0.5025	0.5003	Error

Numerically, $\lim\limits_{x \to 1^-} \dfrac{x-1}{x^2-1} = \dfrac{1}{2}$.

(c) Algebraically, $\lim\limits_{x \to 1^-} \dfrac{x-1}{x^2-1} = \lim\limits_{x \to 1^-} \dfrac{x-1}{(x-1)(x+1)} = \lim\limits_{x \to 1^-} \dfrac{1}{x+1} = \dfrac{1}{2}$.

50. $\lim\limits_{x \to 5^+} \dfrac{5-x}{25-x^2} = 0.1$

(a)



(b)

x	5.1	5.01	5.001	5
$f(x)$	0.099	0.0999	0.09999	Error

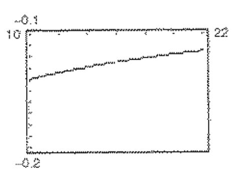
(c) Algebraically,

$$\lim_{x \to 5^+} \frac{5-x}{25-x^2} = \lim_{x \to 5^+} \frac{(5-x)}{(5-x)(5+x)}$$

$$= \lim_{x \to 5^+} \frac{1}{5+x} = \frac{1}{10}.$$

51. $f(x) = \dfrac{4-\sqrt{x}}{x-16}$

(a) Graphically, $\lim\limits_{x \to 16^+} \dfrac{4-\sqrt{x}}{x-16} = -\dfrac{1}{8}$.



(b)

x	16	16.001	16.01	16.1	16.5
$f(x)$	Error	-0.1250	-0.1250	-0.1248	-0.1240

Numerically, $\lim\limits_{x \to 16^+} \dfrac{4-\sqrt{x}}{x-16} = -0.125$.

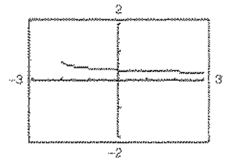
(c) Algebraically,

$$\lim_{x \to 16^+} \frac{4-\sqrt{x}}{x-16} = \lim_{x \to 16^+} \frac{4-\sqrt{x}}{(\sqrt{x}-4)(\sqrt{x}+4)}$$

$$= \lim_{x \to 16^+} \frac{-1}{\sqrt{x}+4} = \frac{-1}{4+4} = -\frac{1}{8}.$$

52. $\lim\limits_{x\to 0^-} \dfrac{\sqrt{x+2}-\sqrt{2}}{x} \approx 0.3536$

(a)



(b)

x	-1.0	-0.1	-0.01	-0.001	0
$f(x)$	0.4142	0.3581	0.3540	0.3536	?

(c) Algebraically,

$$\lim_{x\to 0^-} \frac{\sqrt{x+2}-\sqrt{2}}{x} \cdot \frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}} = \lim_{x\to 0^-} \frac{(x+2)-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)}$$

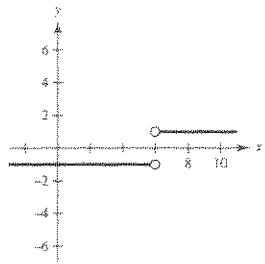
$$= \lim_{x\to 0^-} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.3536.$$

53. $f(x) = \dfrac{|x-6|}{x-6}$

$\lim\limits_{x\to 6^+} f(x) = 1$

$\lim\limits_{x\to 6^-} f(x) = -1$

Limit does not exist.

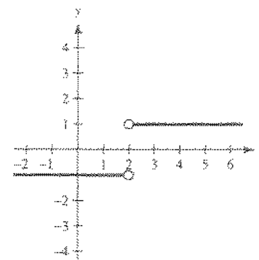


54. $\lim\limits_{x\to 2^-} \dfrac{|x-2|}{x-2} = -1$

$\lim\limits_{x\to 2^+} \dfrac{|x-2|}{x-2} = 1$

$\lim\limits_{x\to 2} \dfrac{|x-2|}{x-2} = 1$

Limit does not exist.

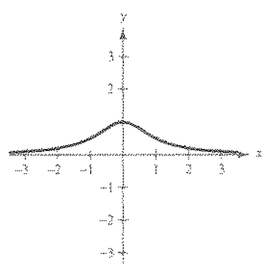


55. $f(x) = \dfrac{1}{x^2+1}$

$\lim\limits_{x\to 1^-} \dfrac{1}{x^2+1} = \lim\limits_{x\to 1^+} \dfrac{1}{x^2+1}$

$= \lim\limits_{x\to 1} \dfrac{1}{x^2+1}$

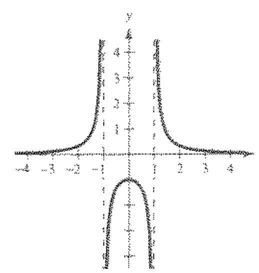
$= \dfrac{1}{2}$



56. $\lim\limits_{x\to 1^-} \dfrac{1}{x^2-1}$ does not exist.

$\lim\limits_{x\to 1^+} \dfrac{1}{x^2-1}$ does not exist.

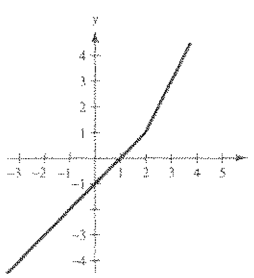
$\lim\limits_{x\to 1} \dfrac{1}{x^2-1}$ does not exist.



57. $\lim\limits_{x\to 2^-} f(x) = 2 - 1 = 1$

$\lim\limits_{x\to 2^+} f(x) = 2(2) - 3 = 1$

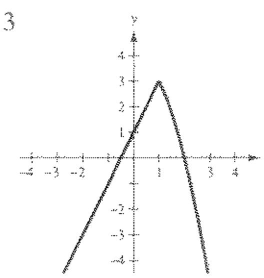
$\lim\limits_{x\to 2} f(x) = 1$



58. $\lim\limits_{x\to 1^-} f(x) = 2(1) + 1 = 3$

$\lim\limits_{x\to 1^+} f(x) = 4 - 1 = 3$

$\lim\limits_{x\to 1} f(x) = 3$

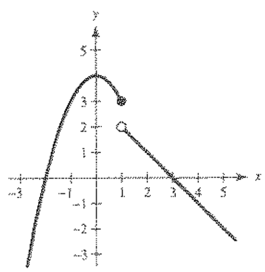


59. $f(x) = \begin{cases} 4 - x^2, & x \le 1 \\ 3 - x, & x > 1 \end{cases}$

$\lim\limits_{x \to 1^-} f(x) = 4 - 1 = 3$

$\lim\limits_{x \to 1^+} f(x) = 3 - 1 = 2$

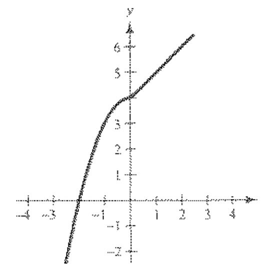
$\lim\limits_{x \to 1} f(x)$ does not exist.



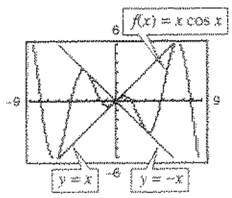
60. $\lim\limits_{x \to 0^-} f(x) = 4 - 0 = 4$

$\lim\limits_{x \to 0^+} f(x) = 0 + 4 = 4$

$\lim\limits_{x \to 0} f(x) = 4$

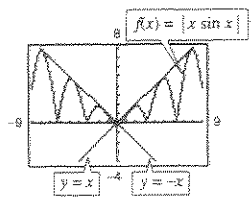


61.



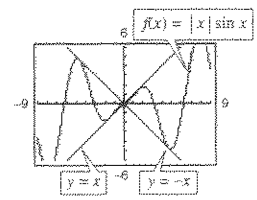
$\lim\limits_{x \to 0} f(x) = 0$

62.



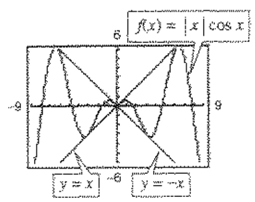
$\lim\limits_{x \to 0} f(x) = 0$

63.



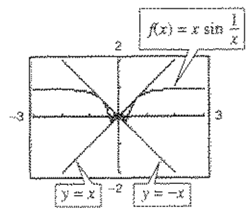
$\lim\limits_{x \to 0} f(x) = 0$

64.



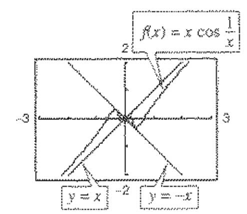
$\lim\limits_{x \to 0} f(x) = 0$

65.



$\lim\limits_{x \to 0} f(x) = 0$

66.



$\lim\limits_{x \to 0} f(x) = 0$

67. (a) Can be evaluated by direct substitution:

$\lim\limits_{x \to 0} x^2 \sin x^2 = 0^2 \sin 0^2 = 0$

(b) Cannot be evaluated by direct substitution:

$$\lim_{x \to 0} \frac{\sin x^2}{x^2} = 1$$

68. (a) Can be evaluated by direct substitution.

$$\lim_{x \to 0} \frac{x}{\cos x} = \frac{0}{\cos 0} = 0$$

(b) Cannot be evaluated by direct substitution.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

(See Section 11.1, Exercise 18.)

69.
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h) - 1 - (3x - 1)}{h}$$
$$= \lim_{h \to 0} \frac{3x + 3h - 1 - 3x + 1}{h}$$
$$= \lim_{h \to 0} \frac{3h}{h} = 3$$

70.
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[5 - 6(x+h)] - (5 - 6x)}{h}$$
$$= \lim_{h \to 0} -\frac{6h}{h} = -6$$

71. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$
$$=\lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

72. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h-2}-\sqrt{x-2}}{h}\cdot\frac{\sqrt{x+h-2}+\sqrt{x-2}}{\sqrt{x+h-2}+\sqrt{x-2}}$$
$$=\lim_{h\to 0}\frac{(x+h-2)-(x-2)}{h\left[\sqrt{x+h-2}+\sqrt{x-2}\right]}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h-2}+\sqrt{x-2}}$$
$$=\frac{1}{2\sqrt{x-2}}$$

73. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{((x+h)^2-3(x+h))-(x^2-3x)}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h}$$
$$=\lim_{h\to 0}\frac{2xh+h^2-3h}{h}$$
$$=\lim_{h\to 0}(2x+h-3)=2x-3$$

74. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[4-2(x+h)-(x+h)^2]-[4-2x-x^2]}{h}$$
$$=\lim_{h\to 0}\frac{4-2x-2h-x^2-2xh-h^2-4+2x+x^2}{h}$$
$$=\lim_{h\to 0}\frac{-2h-2xh-h^2}{h}$$
$$=\lim_{h\to 0}(-2-2x-h)=-2-2x$$

75. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1/(x+h+2)-1/(x+2)}{h}$$
$$=\lim_{h\to 0}\frac{(x+2)-(x+h+2)}{h(x+h+2)(x+2)}$$
$$=\lim_{h\to 0}\frac{-h}{h(x+h+2)(x+2)}$$
$$=\lim_{h\to 0}\frac{-1}{(x+h+2)(x+2)}$$
$$=\frac{-1}{(x+2)^2}$$

76. $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{x+h-1}-\dfrac{1}{x-1}}{h}$

$$= \lim_{h\to 0}\frac{x-1-(x+h-1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)^2}$$

77. $\displaystyle\lim_{t\to 1}\frac{(-16(1)+128)-(-16t^2+128)}{1-t} = \lim_{t\to 1}\frac{16t^2-16}{1-t}$

$$= \lim_{t\to 1}\frac{16(t-1)(t+1)}{1-t}$$

$$= \lim_{t\to 1} -16(t+1)$$

$$= -32\,\frac{\text{ft}}{\text{sec}}$$

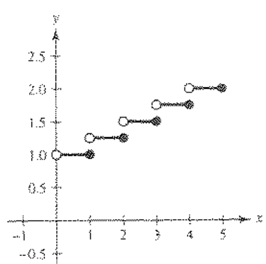
78. $\displaystyle v(2) = \lim_{t\to 2}\frac{s(2)-s(t)}{2-t} = \lim_{t\to 2}\frac{(-64+128)-(-16t^2+128)}{2-t}$

$$= \lim_{t\to 2}\frac{16t^2-64}{2-t} = \lim_{t\to 2}\frac{16(t+2)(t-2)}{2-t}$$

$= \displaystyle\lim_{t\to 2} -16(t+2) = -64$ feet per second

79. $C(t) = 1.00 - 0.25[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.50	1.75	1.75	1.75	1.75	1.75	1.75

$\displaystyle\lim_{t\to 3.5} C(t) = 1.75$

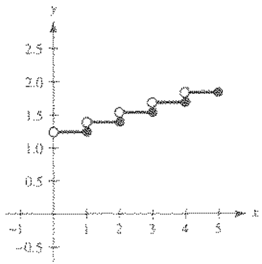
(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.50	1.50	1.50	1.75	1.75	1.75

$\displaystyle\lim_{t\to 3} C(t)$ does not exist. The one-sided limits do not agree.

80. $C(t) = 1.25 - 0.15[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.55	1.70	1.70	1.70	1.70	1.70	1.70

$\lim_{t \to 3.5} C(t) = 1.70$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.4	1.55	1.55	1.55	1.7	1.7	1.7

$\lim_{t \to 3} C(t)$ does not exist. The one-sided limits do not agree.

81. Answers will vary. As $t \to 2$ from the left, $f(t) \to 39.00$. As $t \to 2$ from the right, $f(t) \to 46.80$.

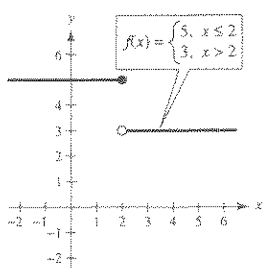
82. Answers will vary. As $x \to 1$ from the left, $f(x) \to 14.40$. As $x \to 1$ from the right, $f(x) \to 18.30$.

83. True

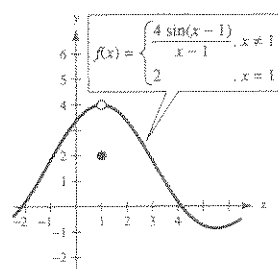
84. False. The value of f at c has no bearing on the limit.

85. Many answers possible

(a)



(b)



86. Answers will vary.

87. Slope of line through $(4, -6)$ and $(3, -4)$:

$$\frac{-6+4}{4-3} = -2$$

Slope of perpendicular line: $\frac{1}{2}$

Equation: $y + 10 = \frac{1}{2}(x - 6)$

$$2y - x + 26 = 0$$

88. Slope between $(3, -3)$ and $(5, -2)$:

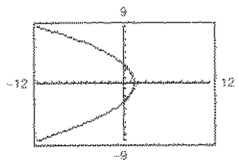
$$\frac{-2-(-3)}{5-3} = \frac{1}{2}$$

Line: $y + 1 = \frac{1}{2}(x - 1)$

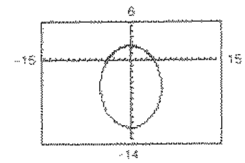
$$2y + 2 = x - 1$$

$$2y - x + 3 = 0$$

89. $r = \dfrac{3}{1 + \cos\theta}$, $e = 1$, Parabola

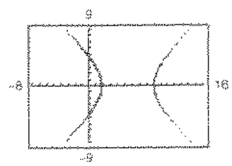


90. $r = \dfrac{12}{3 + 2\sin\theta} = \dfrac{4}{1 + (2/3)\sin\theta}$, $e = \dfrac{2}{3}$, Ellipse

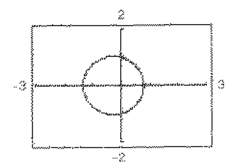


91. $r = \dfrac{9}{2 + 3\cos\theta} = \dfrac{9/2}{1 + (3/2)\cos\theta}$, $e = \dfrac{3}{2}$,

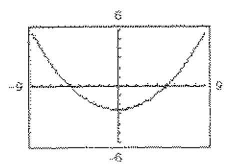
Hyperbola



92. $r = \dfrac{4}{4 + \cos\theta} = \dfrac{1}{1 + (1/4)\cos\theta}$, $e = \dfrac{1}{4}$, Ellipse

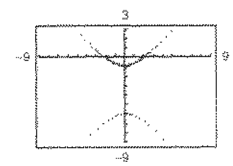


93. $r = \dfrac{5}{1 - \sin\theta}$, $e = 1$, Parabola



94. $r = \dfrac{6}{3 - 4\sin\theta} = \dfrac{2}{1 - (4/3)\sin\theta}$, $e = \dfrac{4}{3}$,

Hyperbola



95. $\langle 7, -2, 3\rangle \cdot \langle -1, 4, 5\rangle = -7 - 8 + 15$

$= 0 \implies$ orthogonal

96. $\langle 5, 5, 0\rangle \cdot \langle 0, 5, 1\rangle = 25 \neq 0$

Not multiples of each other; neither parallel nor orthogonal

97. $-3\langle -4, 3, -6\rangle = \langle 12, -9, 18\rangle \implies$ parallel

98. $\langle 2, -3, 1\rangle \cdot \langle -2, 2, 2\rangle = -8 \neq 0$

Not multiples of each other; neither parallel nor orthogonal

Section 11.3 The Tangent Line Problem

- You should be able to visually approximate the slope of a graph.
- The slope m of the graph of f at the point $(x, f(x))$ is given by

  $$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

  provided this limit exists.
- You should be able to use the limit definition to find the slope of a graph.
- The derivative of f at x is given by

  $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

  provided this limit exists. Notice that this is the same limit as that for the tangent line slope.
- You should be able to use the limit definition to find the derivative of a function.

Vocabulary Check

1. Calculus
2. tangent line
3. secant line
4. difference quotient
5. derivative

1. Slope is 0 at (x, y).
2. Slope is -1 at (x, y).
3. Slope is $\frac{1}{2}$ at (x, y).
4. Slope is -2 at (x, y).

5. $m_{\text{sec}} = \dfrac{g(3 + h) - g(3)}{h} = \dfrac{(3 + h)^2 - 4(3 + h) - (-3)}{h} = \dfrac{h^2 + 2h}{h}$

$m = \lim\limits_{h\to 0} \dfrac{h^2 + 2h}{h} = \lim\limits_{h\to 0} \dfrac{h(h + 2)}{h} = \lim\limits_{h\to 0} (h + 2) = 2$

6. $m_{\text{sec}} = \dfrac{f(3 + h) - f(3)}{h}$

$= \dfrac{10(3 + h) - 2(3 + h)^2 - 12}{h}$

$= \dfrac{-2h - 2h^2}{h} = -2 - 2h, h \neq 0$

$m = \lim\limits_{h\to 0} (-2 - 2h) = -2$

7. $m_{\text{sec}} = \dfrac{g(1 + h) - g(1)}{h}$

$= \dfrac{5 - 2(1 + h) - 3}{h} = \dfrac{-2h}{h}$

$m = \lim\limits_{h\to 0} \dfrac{-2h}{h} = -2$

8. $m_{\text{sec}} = \dfrac{h(-1 + k) - h(-1)}{k} = \dfrac{2(-1 + k) + 5 - 3}{k} = \dfrac{2k}{k}$

$m = \lim\limits_{k\to 0} \dfrac{2k}{k} = 2$

9. $m_{\text{sec}} = \dfrac{g(2 + h) - g(2)}{h} = \dfrac{[4/(2 + h)] - 2}{h} = \dfrac{4 - 2(2 + h)}{(2 + h)h} = \dfrac{-2}{2 + h}, h \neq 0$

$m = \lim\limits_{h\to 0}\left(\dfrac{-2}{2 + h}\right) = -1$

10. $m_{\text{sec}} = \dfrac{g(4 + h) - g(4)}{h} = \dfrac{\dfrac{1}{4 + h - 2} - \dfrac{1}{2}}{h} = \dfrac{\dfrac{1}{2 + h} - \dfrac{1}{2}}{h} = \dfrac{-h}{(2 + h)2h} = \dfrac{-1}{2(2 + h)}, h \neq 0$

$m = \lim\limits_{h\to 0}\left(\dfrac{-1}{2(2 + h)}\right) = -\dfrac{1}{4}$

11. $m_{\text{sec}} = \dfrac{h(9 + k) - h(9)}{k} = \dfrac{\sqrt{9 + k} - 3}{k} \cdot \dfrac{\sqrt{9 + k} + 3}{\sqrt{9 + k} + 3} = \dfrac{(9 + k) - 9}{k[\sqrt{9 + k} + 3]} = \dfrac{1}{\sqrt{9 + k} + 3}, k \neq 0$

$m = \lim\limits_{k\to 0} \dfrac{1}{\sqrt{9 + k} + 3} = \dfrac{1}{6}$

12. $m_{\text{sec}} = \dfrac{h(-1 + k) - h(-1)}{k} = \dfrac{\sqrt{-1 + k + 10} - 3}{k} \cdot \dfrac{\sqrt{k + 9} + 3}{\sqrt{k + 9} + 3} = \dfrac{(k + 9) - 9}{k[\sqrt{k + 9} + 3]} = \dfrac{1}{\sqrt{k + 9} + 3}, k \neq 0$

$m = \lim\limits_{k\to 0} \dfrac{1}{\sqrt{k + 9} + 3} = \dfrac{1}{6}$

13. $m_{\text{sec}} = \dfrac{g(x+h) - g(x)}{h} = \dfrac{4 - (x+h)^2 - (4 - x^2)}{h} = \dfrac{-2xh - h^2}{h} = -2x - h,\ h \neq 0$

$m = \lim\limits_{h \to 0}(-2x - h) = -2x$

(a) At $(0, 4)$, $m = -2(0) = 0$.

(b) At $(-1, 3)$, $m = -2(-1) = 2$.

14. $m_{\text{sec}} = \dfrac{g(x+h) - g(x)}{h} = \dfrac{(x+h)^3 - x^3}{h} = \dfrac{3x^2h + 3xh^2 + h^3}{h}$

$= 3x^2 + 3xh + h^2,\ h \neq 0$

$m = \lim\limits_{h \to 0}(3x^2 + 3xh + h^2) = 3x^2$

(a) At $(1, 1)$, $m = 3(1)^2 = 3$.

(b) At $(-2, -8)$, $m = 3(-2)^2 = 12$.

15. $m_{\text{sec}} = \dfrac{g(x+h) - g(x)}{h} = \dfrac{\dfrac{1}{x+h+4} - \dfrac{1}{x+4}}{h} = \dfrac{(x+4) - (x+4+h)}{(x+h+4)(x+4)(h)}$

$= \dfrac{-h}{(x+h+4)(x+4)h} = \dfrac{-1}{(x+h+4)(x+4)},\ h \neq 0$

$m = \lim\limits_{h \to 0} \dfrac{-1}{(x+h+4)(x+4)} = \dfrac{-1}{(x+4)^2}$

(a) At $\left(0, \dfrac{1}{4}\right)$, $m = \dfrac{-1}{(0+4)^2} = \dfrac{-1}{16}$.

(b) At $\left(-2, \dfrac{1}{2}\right)$, $m = \dfrac{-1}{(-2+4)^2} = \dfrac{-1}{4}$.

16. $m_{\text{sec}} = \dfrac{f(x+h) - f(x)}{h} = \dfrac{\dfrac{1}{x+h+2} - \dfrac{1}{x+2}}{h} = \dfrac{(x+2) - (x+h+2)}{h(x+h+2)(x+2)}$

$= \dfrac{-h}{h(x+h+2)(x+2)}$

$m = \lim\limits_{h \to 0} \dfrac{-h}{h(x+h+2)(x+2)} = \dfrac{-1}{(x+2)^2}$

(a) At $\left(0, \dfrac{1}{2}\right)$, $m = \dfrac{1}{(0+2)^2} = \dfrac{-1}{4}$.

(b) At $(-1, 1)$, $m = \dfrac{1}{(-1+2)^2} = -1$.

17. $m_{\text{sec}} = \dfrac{g(x+h) - g(x)}{h} = \dfrac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \cdot \dfrac{\sqrt{x+h+1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}}$

$= \dfrac{(x+h-1) - (x-1)}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)} = \dfrac{1}{\sqrt{x+h-1} + \sqrt{x-1}},\ h \neq 0$

$m = \lim\limits_{h \to 0}\left(\dfrac{1}{\sqrt{x+h-1} + \sqrt{x-1}}\right) = \dfrac{1}{2\sqrt{x-1}}$

(a) At $(5, 2)$, $m = \dfrac{1}{2\sqrt{5-1}} = \dfrac{1}{4}$.

(b) At $(10, 3)$, $m = \dfrac{1}{2\sqrt{10-1}} = \dfrac{1}{6}$.

18. $m_{\text{sec}} = \dfrac{f(x+h) - f(x)}{h} = \dfrac{\sqrt{x+h-4} - \sqrt{x-4}}{h} \cdot \dfrac{\sqrt{x+h-4} + \sqrt{x-4}}{\sqrt{x+h-4} + \sqrt{x-4}}$

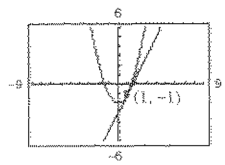
$= \dfrac{(x+h-4) - (x-4)}{h\left[\sqrt{x+h-4} + \sqrt{x-4}\right]}$

$m = \lim\limits_{h \to 0} \dfrac{h}{h\left[\sqrt{x+h-4} + \sqrt{x-4}\right]} = \dfrac{1}{2\sqrt{x-4}}$

(a) At $(5, 1)$, $m = \dfrac{1}{2}$.

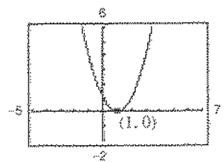
(b) At $(8, 2)$, $m = \dfrac{1}{4}$.

19.



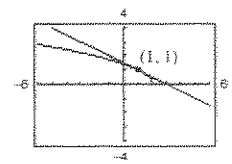
Slope at $(1, -1)$ is 2.

20.



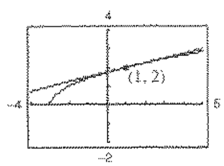
Slope ≈ 0

21.



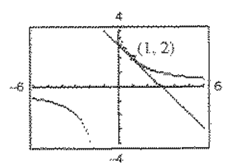
Slope at $(1, 1)$ is $-\frac{1}{2}$.

22.



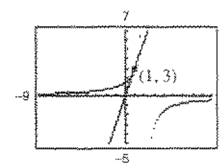
At $(1, 2)$, slope $\approx \frac{1}{4}$.

23.



Slope at $(1, 2)$ is -1.

24.



Slope ≈ 3

25. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{5 - 5}{h} = 0$

26. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(-1) - (-1)}{h} = 0$

27. $g'(x) = \lim\limits_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\left[9 - \frac{1}{3}(x+h)\right] - \left[9 - \frac{1}{3}x\right]}{h}$

$= \lim\limits_{h \to 0} \dfrac{-\frac{1}{3}h}{h} = -\dfrac{1}{3}$

28. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[-5(x+h) + 2] - (-5x + 2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-5h}{h} = -5$

29. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{[4 - 3(x+h)^2] - (4 - 3x^2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-3(x^2 + 2xh + h^2) + 3x^2}{h} = \lim\limits_{h \to 0} \dfrac{-6xh - 3h^2}{h} = \lim\limits_{h \to 0}(-6x - 3h) = -6x$

30. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{[(x+h)^2 - 3(x+h) + 4] - (x^2 - 3x + 4)}{h}$

$= \lim\limits_{h \to 0} \dfrac{x^2 + 2xh + h^2 - 3h - x^2}{h}$

$= \lim\limits_{h \to 0} (2x + h - 3) = 2x - 3$

31. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

$= \lim_{h\to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{(x+h)^2x^2h}$

$= \lim_{h\to 0} \frac{-2x - h}{(x+h)^2x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}$

32. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h}$

$= \lim_{h\to 0} \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{h(x+h)^3\, x^3}$

$= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3}{h(x+h)^3x^3}$

$= \lim_{h\to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3x^3}$

$= \frac{-3x^2}{x^6} = \frac{-3}{x^4}$

33. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\sqrt{x+h-4} - \sqrt{x-4}}{h} \cdot \frac{\sqrt{x+h-4} + \sqrt{x-4}}{\sqrt{x+h-4} + \sqrt{x-4}}$

$= \lim_{h\to 0} \frac{(x+h-4) - (x-4)}{h\left[\sqrt{x+h-4} + \sqrt{x-4}\right]}$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-4} + \sqrt{x-4}}$

$= \frac{1}{2\sqrt{x-4}}$

34. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\sqrt{x+h+8} - \sqrt{x+8}}{h} \cdot \frac{\sqrt{x+h+8} + \sqrt{x+8}}{\sqrt{x+h+8} + \sqrt{x+8}}$

$= \lim_{h\to 0} \frac{(x+h+8) - (x+8)}{h\left[\sqrt{x+h+8} + \sqrt{x+8}\right]}$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+8} + \sqrt{x+8}}$

$= \frac{1}{2\sqrt{x+8}}$

35. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{(x+h+2)} - \frac{1}{x+2}}{h}$

$= \lim_{h\to 0} \frac{(x+2) - (x+h+2)}{h(x+h+2)(x+2)}$

$= \lim_{h\to 0} \frac{-1}{(x+h+2)(x+2)}$

$= \frac{-1}{(x+2)^2}$

36. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{(x+h-5)} - \frac{1}{x-5}}{h}$

$= \lim_{h\to 0} \frac{(x-5) - (x+h-5)}{h(x+h-5)(x-5)}$

$= \lim_{h\to 0} \frac{-1}{(x+h-5)(x-5)}$

$= \frac{-1}{(x-5)^2}$

37. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h-9}} - \frac{1}{\sqrt{x-9}}}{h} \cdot \frac{\frac{1}{\sqrt{x+h-9}} + \frac{1}{\sqrt{x-9}}}{\frac{1}{\sqrt{x+h-9}} + \frac{1}{\sqrt{x-9}}}$

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h-9)} - \frac{1}{(x-9)}}{h\left[\frac{1}{\sqrt{x+h-9}} + \frac{1}{\sqrt{x-9}}\right]} = \lim_{h\to 0} \frac{(x-9)-(x+h-9)}{h(x+h-9)(x-9)\left[\frac{1}{\sqrt{x+h-9}} + \frac{1}{\sqrt{x-9}}\right]}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-9)(x-9)\left[\frac{1}{\sqrt{x+h-9}} + \frac{1}{\sqrt{x-9}}\right]} = \frac{-1}{(x-9)^2\left[\frac{2}{\sqrt{x-9}}\right]} = \frac{-1}{2(x-9)^{3/2}}$$

38. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h+1}} - \frac{1}{\sqrt{x+1}}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{x+1} - \sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \cdot \frac{\sqrt{x+1} + \sqrt{x+h+1}}{\sqrt{x+1} + \sqrt{x+h+1}}$$

$$= \lim_{h\to 0} \frac{(x+1)-(x+h+1)}{h\sqrt{x+h+1}\sqrt{x+1}\left[\sqrt{x+1} + \sqrt{x+h+1}\right]}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h+1}\sqrt{x+1}\left[\sqrt{x+1} + \sqrt{x+h+1}\right]}$$

$$= \frac{-1}{\sqrt{x+1}\sqrt{x+1}\left[2\sqrt{x+1}\right]}$$

$$= \frac{-1}{2(x+1)^{3/2}}$$

39. (a) $m_{\text{sec}} = \frac{f(2+h) - f(2)}{h}$

$$= \frac{(2+h)^2 - 1 - 3}{h}$$

$$= \frac{4 + 4h + h^2 - 4}{h}$$

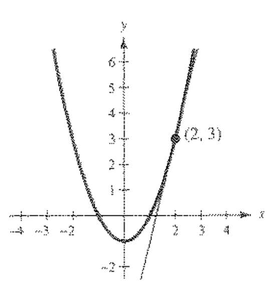
$$= 4 + h,\ h \neq 0$$

$$m = \lim_{h\to 0}(4+h) = 4$$

(b) $y - 3 = 4(x - 2)$

$$y = 4x - 5$$

(c)



40. (a) $m_{\text{sec}} = \frac{f(1+h) - f(1)}{h}$

$$= \frac{4 - (1+h)^2 - 3}{h}$$

$$= \frac{4 - (1 + 2h + h^2) - 3}{h}$$

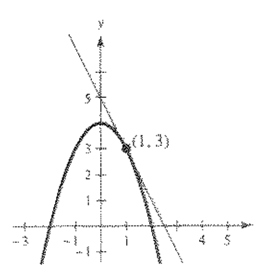
$$= \frac{-h^2 - 2h}{h} = -h - 2,\ h \neq 0$$

$$m = \lim_{h\to 0}(-h-2) = -2$$

(b) $y - 3 = -2(x - 1)$

$$y = -2x + 5$$

(c)



41. (a) $m_{sec} = \dfrac{f(1+h) - f(1)}{h}$

$= \dfrac{(1+h)^3 - 2(1+h) - (-1)}{h}$

$= \dfrac{1 + 3h + 3h^2 + h^3 - 2 - 2h + 1}{h}$

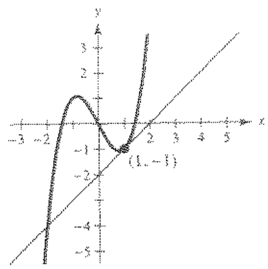
$= \dfrac{h^3 + 3h^2 + h}{h} = h^2 + 3h + 1,\ h \neq 0$

$m = \lim_{h \to 0} (h^2 + 3h + 1) = 1$

(b) $y - (-1) = 1(x - 1)$

$y = x - 2$

(c)



42. (a) $m_{sec} = \dfrac{f(2+h) - f(2)}{h}$

$= \dfrac{(2+h)^3 - (2+h) - 6}{h}$

$= \dfrac{h^3 + 6h^2 + 12h + 8 - 2 - h - 6}{h}$

$= \dfrac{h^3 + 6h^2 + 11h}{h}$

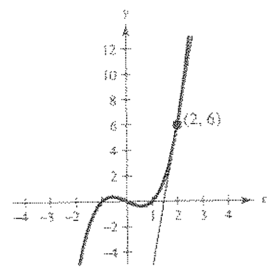
$= h^2 + 6h + 11,\ h \neq 0$

$m = \lim_{h \to 0} (h^2 + 6h + 11) = 11$

(b) $y - 6 = 11(x - 2)$

$y = 11x - 16$

(c)



43. (a) $m_{sec} = \dfrac{f(3+h) - f(3)}{h} = \dfrac{\sqrt{3+h+1} - 2}{h}$

$= \dfrac{\sqrt{4+h} - 2}{h} \cdot \dfrac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$

$= \dfrac{4 + h - 4}{h\left[\sqrt{4+h} + 2\right]}$

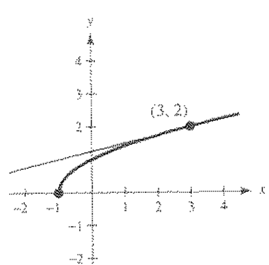
$= \dfrac{1}{\sqrt{4+h} + 2}$

$m = \lim_{h \to 0} \dfrac{1}{\sqrt{4+h} + 2} = \dfrac{1}{4}$

(b) $y - 2 = \dfrac{1}{4}(x - 3)$

$y = \dfrac{1}{4}x + \dfrac{5}{4}$

(c)



44. (a) $m_{sec} = \dfrac{f(3+h) - f(3)}{h} = \dfrac{\sqrt{3+h-2} - 1}{h}$

$= \dfrac{\sqrt{1+h} - 1}{h} \cdot \dfrac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$

$= \dfrac{(1+h) - 1}{h\left[\sqrt{1+h} + 1\right]}$

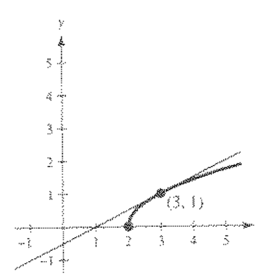
$= \dfrac{1}{\sqrt{1+h} + 1},\ h \neq 0$

$m = \lim_{h \to 0} \dfrac{1}{\sqrt{1+h} + 1} = \dfrac{1}{2}$

(b) $y - 1 = \dfrac{1}{2}(x - 3)$

$y = \dfrac{1}{2}x - \dfrac{1}{2}$

(c)



45. (a) $m_{sec} = \dfrac{f(-4+h) - f(-4)}{h}$

$$= \frac{\dfrac{1}{-4+h+5} - 1}{h}$$

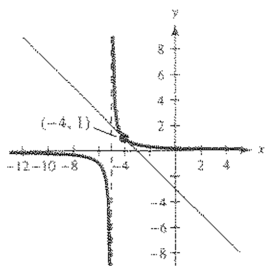
$$= \frac{1 + 4 - h - 5}{h(-4+h+5)} = \frac{-1}{h+1},\ h \neq 0$$

$$m = \lim_{h \to 0} \frac{-1}{h+1} = -1$$

(b) $y - 1 = -1(x + 4)$

$y = -x - 3$

(c)



46. (a) $m_{sec} = \dfrac{f(4+h) - f(4)}{h}$

$$= \frac{\dfrac{1}{4+h-3} - 1}{h}$$

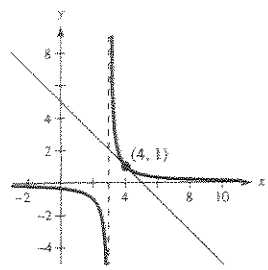
$$= \frac{1 - 4 - h + 3}{h(4+h-3)} = \frac{-1}{h+1},\ h \neq 0$$

$$m = \lim_{h \to 0} \frac{-1}{h+1} = -1$$

(b) $y - 1 = -1(x - 4)$

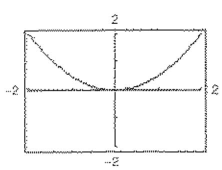
$y = -x + 5$

(c)



47.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	2	1.125	0.5	0.125	0	0.125	0.5	1.125	2
$f'(x)$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2



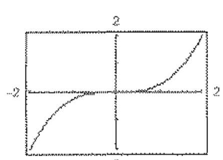
$f(x) = \frac{1}{2}x^2$

$f'(x) = x$

They appear to be the same.

48.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	-0.844	-0.25	-0.031	0	0.031	0.25	0.844	2
$f'(x)$	3	1.688	0.75	0.188	0	0.188	0.75	1.688	3



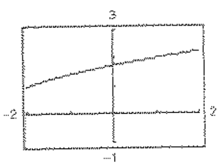
$f(x) = \frac{1}{4}x^3$

$f'(x) = \frac{3}{4}x^2$

They appear to be the same.

49.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	1	1.225	1.414	1.581	1.732	1.871	2	2.121	2.236
$f'(x)$	0.5	0.408	0.354	0.316	0.289	0.267	0.25	0.236	0.224



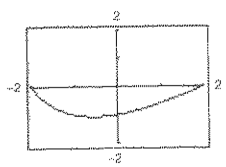
$$f(x) = \sqrt{x+3}$$

$$f'(x) = \frac{1}{2\sqrt{x+3}}$$

They appear to be the same.

50.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	0	-0.7	-1	-1.071	-1	-0.833	-0.6	-0.318	0
$f'(x)$	-2	-0.92	-0.333	0.020	0.25	0.407	0.52	0.603	0.667



$$f(x) = \frac{x^2 - 4}{x + 4}$$

$$f'(x) = \frac{x^2 + 8x + 4}{(x+4)^2}$$

They appear to be the same.

51. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)^2 - 4(x+h) + 3] - [x^2 - 4x + 3]}{h}$

$= \lim_{h\to 0} \dfrac{(x^2 + 2xh + h^2 - 4x - 4h + 3) - (x^2 - 4x + 3)}{h}$

$= \lim_{h\to 0} \dfrac{2xh + h^2 - 4h}{h} = \lim_{h\to 0} 2x + h - 4 = 2x - 4$

$f'(x) = 0 = 2x - 4 \implies x = 2$

f has a horizontal tangent at $(2, -1)$.

52. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \dfrac{(x+h)^2 - 6(x+h) + 4 - (x^2 - 6x + 4)}{h}$

$= \lim_{h\to 0} \dfrac{x^2 + 2xh + h^2 - 6x - 6h - x^2 + 6x}{h}$

$= \lim_{h\to 0} \dfrac{2xh + h^2 - 6hh}{h} = 2x - 6$

$f'(x) = 2x = 6$

$f'(x) = 0 \implies x = 3$

f has a horizontal tangent at $(3, -5)$.

53. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{[3(x+h)^3 - 9(x+h)] - [3x^3 - 9x]}{h}$

$= \lim_{h\to 0} \dfrac{3(x^3 + 3x^2h + 3xh^2 + h^3) - 9x - 9h - 3x^3 + 9x}{h}$

$= \lim_{h\to 0} \dfrac{9x^2h + 9xh^2 + 3h^3 - 9h}{h}$

$= \lim_{h\to 0} (9x^2 + 9xh + 3h - 9) = 9x^2 - 9$

$f'(x) = 0 = 9x^2 - 9 = 9(x+1)(x-1) \implies x = \pm 1$

Horizontal tangents at $(1, -6)$ and $(-1, 6)$

54. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^3 + 3(x+h) - (x^3 + 3x)}{h}$

$= \lim_{h\to 0} \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h - x^3 - 3x}{h}$

$= \lim_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3 + 3h}{h}$

$= \lim_{h\to 0} (3x^2 + 3xh + h^2 + 3) = 3x^2 + 3$

$f'(x) = 3x^2 + 3 = 0,$ Impossible

No horizontal tangents

55. $f'(x) = 4x^3 - 4x = 0$

$4x(x-1)(x+1) = 0$

$x = 0, 1, -1$

$(0, 0), (1, -1), (-1, -1)$

56. $f'(x) = 12x^3 + 12x^2$

$f'(x) = 0 = 12x^3 + 12x^2 = 12x^2(x+1) \implies 0, -1$

f has horizontal tangents at $(0, 0)$ and $(-1, -1)$.

57. $f'(x) = -2\sin x + 1 = 0$

$\sin x = \dfrac{1}{2}$

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\left(\dfrac{\pi}{6}, \sqrt{3} + \dfrac{\pi}{6}\right), \left(\dfrac{5\pi}{6}, \dfrac{5\pi}{6} - \sqrt{3}\right)$

58. $f'(x) = 1 - 2\cos x = 0$

$\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$\left(\dfrac{\pi}{3}, \dfrac{\pi}{3} - \sqrt{3}\right), \left(\dfrac{5\pi}{3}, \dfrac{5\pi}{3} + \sqrt{3}\right)$

59. $f'(x) = x^2e^x + 2xe^x = 0$

$xe^x(x + 2) = 0$

$x = 0, -2$

$(0, 0), (-2, 4e^{-2})$

60. $f'(x) = e^{-x} - xe^{-x} = 0$

$e^{-x}(1 - x) = 0$

$x = 1$

$(1, e^{-1})$

61. $f'(x) = \ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1}$

$(e^{-1}, -e^{-1})$

62. $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$

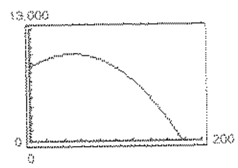
$1 - \ln x = 0$

$x = e$

$\left(e, \dfrac{1}{e}\right)$

63. (a) $P(t) = -0.63t^2 + 63.3t + 8448$

(b)



(c) Using the limit definition,

$P'(t) = -1.26t + 63.3$

$P'(20) = -1.26(20) + 63.3 = 38.1.$

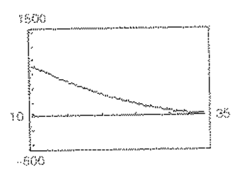
$P'(20) \approx 38$

At time 2020, the population is increasing at approximately 38,000 per year.

(d) Answers will vary.

64. (a) $N = 1.04p^2 - 81.50p + 1613.31$

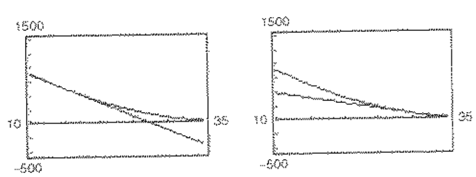
(b)



Slope $= -50.3$ for $p = 15$.

Slope $= -19.1$ for $p = 30$.

(c)



(d) The rate of decrease in sales decreases as the price increases.

65. (a) $V = \dfrac{4}{3}\pi r^3$

$$V'(r) = \lim_{h \to 0} \frac{V(r + h) - V(r)}{h}$$

$$= \lim_{h \to 0} \frac{(4/3)\pi(r + h)^3 - (4/3)\pi r^3}{h}$$

$$= \lim_{h \to 0} \left(\frac{4}{3}\pi\right)\frac{r^3 + 3r^2h + 3rh^2 + h^3 - r^3}{h}$$

$$= \lim_{h \to 0} \frac{4}{3}\pi(3r^2 + 3rh + h^2) = 4\pi r^2$$

(b) $V'(4) = 4\pi(4)^2 \approx 201.06$

(c) Cubic inches per inch; Answers will vary.

66. $S(r) = 4\pi r^2$

(a) Using the limit definition, $S'(r) = 8\pi r$.

(b) $S'(2) = 8\pi(2) = 16\pi \approx 50.27$

(c) Square millimeters per millimeter

67. $s(t) = -16t^2 + 64t + 80$

(a) Using the limit definition, $s'(t) = -32t + 64$.

(b) $s(0) = 80$, $s(3) = 128$

$$\text{Average rate of change} = \frac{128 - 80}{3} = \frac{48}{3} = 16 \text{ ft/sec}$$

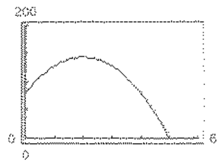
(c) $s'(t) = -32t + 64 = 0 \implies t = 2$ seconds

Answers will vary.

(d) $s(t) = -16t^2 + 64t + 80 = 0 \implies t = 5$ seconds

$s'(5) = -32(5) + 64 = -96$ ft/sec

(e)



68. $s(t) = -16t^2 + 120$

(a) Using the limit definition, $s'(t) = -32t$.

(b) $\text{Average} = \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{56 - 120}{2} = -32$ ft/sec

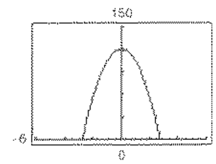
(c) $s(t) = -16t^2 + 120 = 0 \implies t = \sqrt{7.5}$

$s'(\sqrt{7.5}) = -32\sqrt{7.5} \approx -87.64$ ft/sec

(d) $-32t = -60$

$t = \dfrac{60}{32} = 1.875$ seconds

(e)



69. True. The slope is $2x$, which is different for all x.

70. False. For example, the tangent line to $y = x^3$ at $(1, 1)$ intersects the curve at $(-2, -8)$.

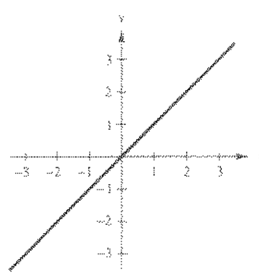
71. Matches (b).
(Derivative is always positive, but decreasing.)

72. Matches (a).
(Derivative approaches $-\infty$ when x approaches 0.)

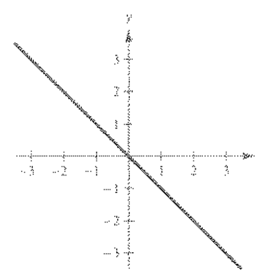
73. Matches (d).
(Derivative is -1 for $x < 0$, 1 for $x > 0$.)

74. Matches (c).
(Derivative decreases until origin, then increases.)

75. Answers will vary.



76. Answers not unique

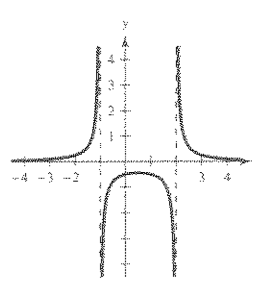


77. $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x + 1)(x - 2)}$

Intercept: $\left(0, -\dfrac{1}{2}\right)$

Vertical asymptotes: $x = -1$, $x = 2$

Horizontal asymptote: $y = 0$

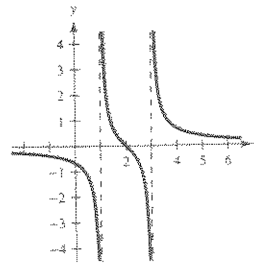


78. $f(x) = \dfrac{x-2}{x^2-4x+3} = \dfrac{x-2}{(x-3)(x-1)}$

Intercepts: $(2, 0), \left(0, -\dfrac{2}{3}\right)$

Vertical asymptotes: $x = 1, x = 3$

Horizontal asymptote: $y = 0$

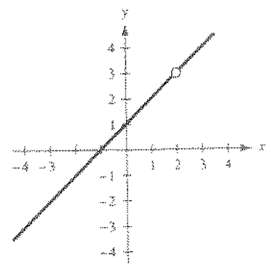


79. $f(x) = \dfrac{x^2-x-2}{x-2} = \dfrac{(x-2)(x+1)}{x-2} = x + 1,\ x \neq 2$

Line with hole at $(2, 3)$

Intercepts: $(0, 1), (-1, 0)$

Slant asymptote: $y = x + 1$

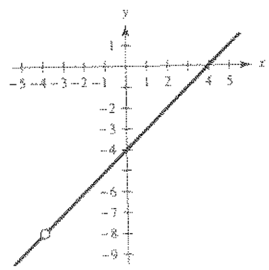


80. $f(x) = \dfrac{x^2-16}{x+4} = \dfrac{(x+4)(x-4)}{x+4} = x - 4,\ x \neq -4$

Line with hole at $(-4, -8)$

Intercepts: $(0, -4), (4, 0)$

Slant asymptote: $y = x - 4$



81. $\langle 1, 1, 1\rangle \times \langle 2, 1, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$

$= \langle -2, 3, -1\rangle$

82. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 & 0 & 6 \\ 7 & 0 & 0 \end{vmatrix} = \langle 0, 42, 0\rangle$

83. $\langle -4, 10, 0\rangle \times \langle 4, -1, 0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 10 & 0 \\ 4 & -1 & 0 \end{vmatrix}$

$= \langle 0, 0, -36\rangle$

84. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -7 & 14 \\ -1 & 8 & 4 \end{vmatrix} = \langle -140, -46, 57\rangle$

Section 11.4 Limits at Infinity and Limits of Sequences

- The limit at infinity $\lim\limits_{x\to\infty} f(x) = L$ means that $f(x)$ get arbitrarily close to L as x increases without bound.
- Similarly, the limit at infinity $\lim\limits_{x\to-\infty} f(x) = L$ means that $f(x)$ get arbitrarily close to L as x decreases without bound.
- You should be able to calculate limits at infinity, especially those arising from rational functions.
- Limits of functions can be used to evaluate limits of sequences. If f is a function such that $\lim\limits_{x\to\infty} f(x) = L$ and if a_n is a sequence such that $f(n) = a_n$, then $\lim\limits_{n\to\infty} a_n = L$.

Vocabulary Check

1. limit, infinity
2. converge
3. diverge

1. Intercept: $(0, 0)$

Horizontal asymptote: $y = 4$

Matches (c).

2. Horizontal asymptote: $y = 1$

Matches (a).

3. Horizontal asymptote: $y = 4$

Vertical asymptote: $x = 0$

Matches (d).

4. $f(x) = x + \dfrac{1}{x}$

No horizontal asymptote

Matches (b).

5. Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$

Matches (f).

6. Vertical asymptote: $x = 2$

Horizontal asymptote: $y = 2$

Matches (g).

7. Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -2$

Matches (h).

8. Vertical asymptote: $x = \pm 2$

Horizontal asymptote: $y = -4$

Matches (e).

9. $\displaystyle\lim_{x\to\infty} \frac{3}{x^2} = 0$

10. $\displaystyle\lim_{x\to\infty} \frac{5}{2x} = 0$

11. $\displaystyle\lim_{x\to\infty} \frac{3+x}{3-x} = -1$

12. $\displaystyle\lim_{x\to\infty} \frac{2-7x}{2+3x} = -\frac{7}{3}$

13. $\displaystyle\lim_{x\to-\infty} \frac{5x-2}{6x+1} = \frac{5}{6}$

14. $\displaystyle\lim_{x\to-\infty} \frac{5-3x}{x+4} = -3$

15. $\displaystyle\lim_{x\to-\infty} \frac{4x^2-3}{2-x^2} = \frac{4}{-1} = -4$

16. $\displaystyle\lim_{x\to-\infty} \frac{x^2+3}{5x^2-4} = \frac{1}{5}$

17. $\displaystyle\lim_{t\to\infty} \frac{t^2}{t+3}$ does not exist.

18. $\displaystyle\lim_{y\to\infty} \frac{4y^4}{y^2+3}$ does not exist.

19. $\displaystyle\lim_{t\to\infty} \frac{4t^2+3t-1}{3t^2+2t-5} = \frac{4}{3}$

20. $\displaystyle\lim_{x\to\infty} \frac{5-6x-3x^2}{2x^2+x+4} = -\frac{3}{2}$

21. $\displaystyle\lim_{y\to-\infty} \frac{3+8y-4y^2}{3-y-2y^2} = \frac{-4}{-2} = 2$

22. $\displaystyle\lim_{t\to-\infty} \frac{t^2+9t-10}{2+4t-3t^2} = \frac{1}{-3} = -\frac{1}{3}$

23. $\displaystyle\lim_{x\to-\infty} \frac{-(x^2+3)}{(2-x)^2} = \lim_{x\to-\infty} \frac{-x^2-3}{x^2-4x+4} = -1$

24. $\displaystyle\lim_{x\to\infty} \frac{2x^2-6}{(x-1)^2} = \lim_{x\to\infty} \frac{2x^2-6}{x^2-2x+1} = 2$

25. $\displaystyle\lim_{x\to-\infty} \left[\frac{x}{(x+1)^2} - 4\right] = 0 - 4 = -4$

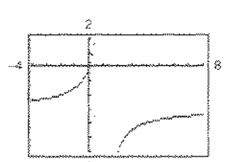
26. $\displaystyle\lim_{x\to\infty} \left[7 + \frac{2x^2}{(x+3)^2}\right] = 7 + 2 = 9$

27. $\displaystyle\lim_{t\to\infty} \left(\frac{1}{3t^2} - \frac{5t}{t+2}\right) = 0 - 5 = -5$

28. $\displaystyle\lim_{x\to\infty} \left[\frac{x}{2x+1} + \frac{3x^2}{(x-3)^2}\right] = \frac{1}{2} + 3 = \frac{7}{2}$

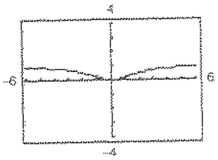
29. $y = \dfrac{3x}{1-x}$

Horizontal asymptote: $y = -3$



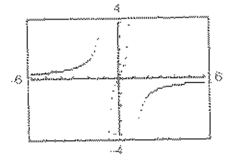
30. $y = \dfrac{x^2}{x^2 + 4}$

Horizontal asymptote: $y = 1$



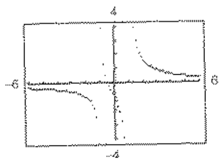
31. $y = \dfrac{2x}{1 - x^2}$

Horizontal asymptote: $y = 0$



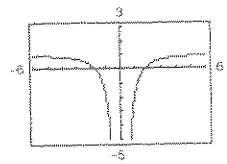
32. $y = \dfrac{2x + 1}{x^2 - 1}$

Horizontal asymptote: $y = 0$



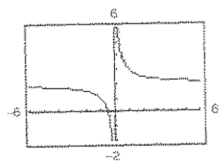
33. $y = 1 - \dfrac{3}{x^2}$

Horizontal asymptote: $y = 1$



34. $y = 2 + \dfrac{1}{x}$

Horizontal asymptote: $y = 2$



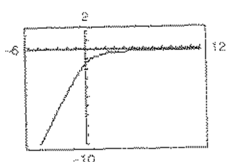
35. $f(x) = x - \sqrt{x^2 + 2}$

(a)

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-0.7321	-0.0995	-0.0100	-0.0010	-1.0×10^{-4}	-1.0×10^{-5}	-1.0×10^{-6}

$\lim_{x\to\infty} f(x) = 0$

(b)



$\lim_{x\to\infty} f(x) = 0$

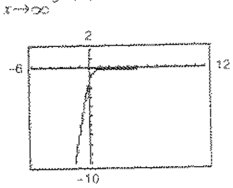
36. $f(x) = 3x - \sqrt{9x^2 + 1}$

(a)

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-0.1623	-0.0167	-0.0017	-1.67×10^{-4}	-1.67×10^{-5}	-1.67×10^{-6}	-1.67×10^{-7}

$\lim_{x\to\infty} f(x) = 0$

(b)



$\lim_{x\to\infty} f(x) = 0$

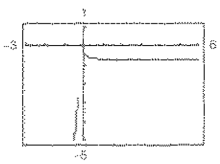
37. $f(x) = 3(2x - \sqrt{4x^2 + x})$

(a)

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-0.7082	-0.7454	-0.7495	-0.74995	-0.749995	-0.7499995	-0.75

$\lim_{x\to\infty} f(x) = -0.75$

(b)



$\lim_{x\to\infty} f(x) = -0.75$

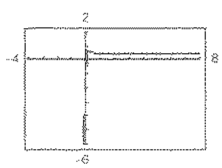
38. $f(x) = 4(4x - \sqrt{16x^2 - x})$

(a)

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.5081	0.5008	0.5001	0.5000	0.5000	0.5000	0.5000

$\lim_{x\to\infty} f(x) = 0.5$

(b)



$\lim_{x\to\infty} f(x) = 0.5$

39. $a_n = \dfrac{n+1}{n^2+1}$

$1, \frac{3}{5}, \frac{2}{5}, \frac{5}{17}, \frac{3}{13}$

$\lim_{n\to\infty} a_n = 0$

40. $a_n = \dfrac{n}{n^2+1}$

$\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}$

$\lim_{n\to\infty} \dfrac{n}{n^2+1} = 0$

41. $a_n = \dfrac{n}{2n+1}$

$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$

$\lim_{n\to\infty} a_n = \dfrac{1}{2}$

42. $a_n = \dfrac{4n-1}{n+3}$

$\frac{3}{4}, \frac{7}{5}, \frac{11}{6}, \frac{15}{7}, \frac{19}{8}$

$\lim_{n\to\infty} \dfrac{4n-1}{n+3} = 4$

43. $a_n = \dfrac{n^2}{3n+2}$

$\frac{1}{5}, \frac{1}{2}, \frac{9}{11}, \frac{8}{7}, \frac{25}{17}$

$\lim_{n\to\infty} \dfrac{n^2}{3n+2}$ does not exist.

44. $a_n = \dfrac{4n^2+1}{2n}$

$\frac{5}{2}, \frac{17}{4}, \frac{37}{6}, \frac{65}{8}, \frac{101}{10}$

$\lim_{n\to\infty} \dfrac{4n^2+1}{2n}$ does not exist.

45. $a_n = \dfrac{(n+1)!}{n!}$

2, 3, 4, 5, 6

$\lim_{n\to\infty} \dfrac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1)$ does not exist.

46. $a_n = \dfrac{(3n-1)!}{(3n+1)!} = \dfrac{1}{(3n+1)(3n)}$

$\frac{1}{12}, \frac{1}{42}, \frac{1}{90}, \frac{1}{156}, \frac{1}{240}$

$\lim_{n\to\infty} \dfrac{(3n-1)!}{(3n+1)!} = 0$

47. $a_n = \dfrac{(-1)^n}{n}$

$-1, \dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, -\dfrac{1}{5}$

$\lim_{n\to\infty} \dfrac{(-1)^n}{n} = 0$

48. $a_n = \dfrac{(-1)^{n+1}}{n^2}$

$1, -\dfrac{1}{4}, \dfrac{1}{9}, -\dfrac{1}{16}, \dfrac{1}{25}$

$\lim_{n\to\infty} \dfrac{(-1)^{n+1}}{n^2} = 0$

49.

n	10^0	10^1	10^2	10^3	10^4	10^5	10^6
a_n	2	1.55	1.505	1.5005	1.5001	1.500	1.500

$\lim_{n\to\infty} a_n = 1.5$

$$a_n = \frac{1}{n}\left(n + \frac{1}{n}\left[\frac{n(n+1)}{2}\right]\right) = 1 + \frac{1}{n^2}\left[\frac{n^2+n}{2}\right] = 1 + \frac{n^2+n}{2n^2}$$

$$\lim_{n\to\infty} a_n = 1 + \frac{1}{2} = \frac{3}{2}$$

50.

n	10^0	10^1	10^2	10^3	10^4	10^5	10^6
a_n	20	12.8	12.08	12.008	12.0008	12.0001	12.0000

$\lim_{n\to\infty} a_n = 12.0$

$$a_n = \frac{4}{n}\left(n + \frac{4}{n}\left[\frac{n(n+1)}{2}\right]\right) = 4 + \frac{16}{n^2}\left[\frac{n^2+n}{2}\right] = 4 + \frac{8(n^2+n)}{n^2}$$

$$\lim_{n\to\infty} a_n = 4 + 8 = 12$$

51.

n	10^0	10^1	10^2	10^3	10^4	10^5	10^6
a_n	16	6.16	5.4136	5.3413	5.3341	5.3334	5.3333

$\lim_{n\to\infty} a_n = 5.33$

$$a_n = \frac{16}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \left(\frac{8}{3}\right)\frac{n(n+1)(2n+1)}{n^3}$$

$$\lim_{n\to\infty} a_n = \frac{16}{3}$$

52.

n	10^0	10^1	10^2	10^3	10^4	10^5	10^6
a_n	1	0.7975	0.7550	0.7505	0.7501	0.7500	0.7500

$\lim_{n\to\infty} a_n = 0.75$

$$a_n = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2+n}{n^2} - \left(\frac{1}{4}\right)\frac{(n^2+n)^2}{n^4}$$

$$\lim_{n\to\infty} a_n = 1 - \frac{1}{4} = \frac{3}{4}$$

53. (a) Average cost $= \overline{C} = \dfrac{C}{x} = 13.50 + \dfrac{45{,}750}{x}$

(b) $\overline{C}(100) = \$471$

$\overline{C}(1000) = \$59.25$

(c) $\lim\limits_{x\to\infty} C(x) = 13.50$

As more units are produced, the fixed costs (45,750) become less dominant.

54. $C = 1.25x + 10{,}500$

(a) Average cost $= \overline{C} = \dfrac{C}{x} = 1.25 + \dfrac{10{,}500}{x}$

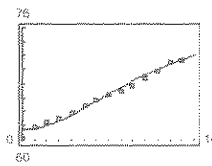
(b) $\overline{C}\,(100) = \$106.25$

$\overline{C}\,(1000) = \$11.75$

(c) As $x \to \infty$, $\overline{C} \to \$1.25$.

As the number of tons gets very large, the average cost approaches \$1.25 per ton.

55. (a)



(b) For 2004, $t = 14$ and $E(14) \approx 72.0$ million.

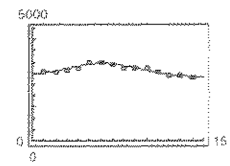
For 2008, $t = 18$ and $E(18) \approx 73.8$ million.

(c) $\lim\limits_{t\to\infty} E(t) = \dfrac{0.702}{0.009} = 78$

The enrollment approaches 78 million.

(d) Answers will vary.

56. (a)



(b) For 2006, $t = 16$ and $N \approx 2772$ thousand.

For 2010, $t = 20$ and $N \approx 2696$ thousand.

(c) $\lim\limits_{t\to\infty} N(t) = \dfrac{40.8189}{0.0157} \approx 2600$

The number of injuries approaches 2600 thousand.

(d) Answers will vary.

57. False. $f(x) = \dfrac{x^2+1}{1}$ does not have a horizontal asymptote.

58. False. The limit does not exist.

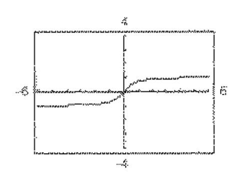
59. True

60. False

61. For example, let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^2}$.

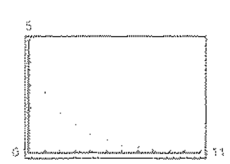
Then, $\lim\limits_{x\to 0} \dfrac{1}{x^2}$ increases without bound, but $\lim\limits_{x\to 0}\,[f(x) - g(x)] = 0$.

62.

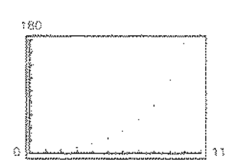


Two horizontal asymptotes: $y = \pm 1$

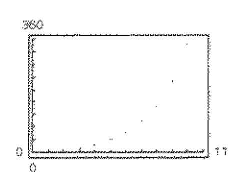
63. Converges to 0



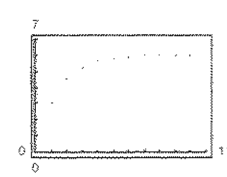
64. Diverges



65. Diverges



66. Converges to 6



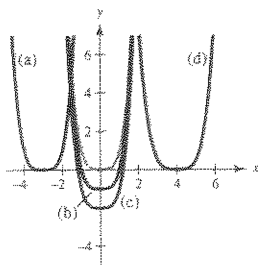
67. $y = x^4$

(a) $f(x) = (x + 3)^4$

(b) $f(x) = x^4 - 1$

(c) $f(x) = -2 + x^4$

(d) $f(x) = \frac{1}{2}(x - 4)^4$



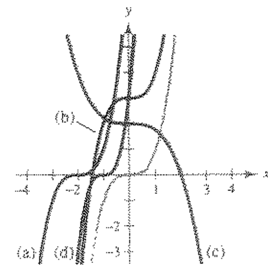
68. $y = x^3$

(a) $f(x) = (x + 2)^3$

(b) $f(x) = 3 + x^3$

(c) $f(x) = 2 - \frac{1}{4}x^3$

(d) $f(x) = 3(x + 1)^3$



69.

$$\begin{array}{r}
x^2 + 2x + 1 \\
x^2 - 4 \overline{\smash{)}\, x^4 + 2x^3 - 3x^2 - 8x - 4} \\
\underline{x^4 \qquad\quad - 4x^2} \qquad\qquad\quad \\
2x^3 + x^2 \qquad\qquad \\
\underline{2x^3 \qquad\quad - 8x} \quad\;\; \\
x^2 \qquad - 4
\end{array}$$

$x^4 + 2x^3 - 3x^2 - 8x - 4 = (x^2 - 4)(x^2 + 2x + 1)$

70.

$$\begin{array}{r}
2x^3 + 4x^2 - 2x - 8 \\
x^2 - 2x + 1 \overline{\smash{)}\, 2x^5 \qquad\quad - 8x^3 \qquad\quad + 4x - 1} \\
\underline{2x^5 - 4x^4 + 2x^3} \qquad\qquad\qquad\qquad \\
4x^4 - 10x^3 \qquad\qquad\qquad\quad \\
\underline{4x^4 - 8x^3 + 4x^2} \qquad\qquad\quad \\
-2x^3 - 4x^2 + 4x \qquad\quad \\
\underline{-2x^3 + 4x^2 - 2x} \qquad\quad \\
-8x^2 + 6x - 1 \\
\underline{-8x^2 + 16x - 8} \\
-10x + 7
\end{array}$$

$$\frac{2x^5 - 8x^3 + 4x - 1}{x^2 - 2x + 1} = 2x^3 + 4x^2 - 2x - 8 + \frac{-10x + 7}{x^2 - 2x + 1}$$

71.

$$\begin{array}{r}
x^3 + 5x^2 \qquad - 3 \\
3x + 2 \overline{\smash{)}\, 3x^4 + 17x^3 + 10x^2 - 9x - 8} \\
\underline{3x^4 + 2x^3} \qquad\qquad\qquad\qquad\quad \\
15x^3 + 10x^2 \qquad\qquad\quad \\
\underline{15x^3 + 10x^2} \qquad\qquad\quad \\
-9x - 8 \\
\underline{-9x - 6} \\
-2
\end{array}$$

$$\frac{3x^4 + 17x^3 + 10x^2 - 9x - 8}{3x + 2} = x^3 + 5x^2 - 3 + \frac{-2}{3x + 2}$$

72.

$$\begin{array}{r}
2x^2 + 11x + 14 \\
5x - 2 \overline{\smash{\big)}\, 10x^3 + 51x^2 + 48x - 28} \\
\underline{10x^3 - 4x^2} \qquad\qquad\quad \\
55x^2 + 48x \qquad\quad \\
\underline{55x^2 - 22x} \qquad\quad \\
70x - 28 \\
\underline{70x - 28}
\end{array}$$

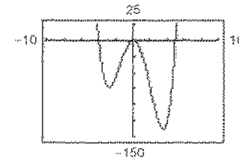
$$\frac{10x^3 + 51x^2 + 48x - 28}{5x - 2} = 2x^2 + 11x + 14, \quad x \neq \frac{2}{5}$$

73. $f(x) = x^4 - x^3 - 20x^2$

$= x^2(x^2 - x - 20)$

$= x^2(x - 5)(x + 4)$

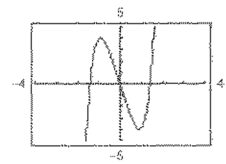
Real zeros: $0, 0, 5, -4$



74. $x^5 + x^3 - 6x = x(x^4 + x^2 - 6)$

$= x(x^2 + 3)(x^2 - 2)$

Real zeros: $0, \pm\sqrt{2}$

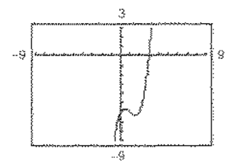


75. $f(x) = x^3 - 3x^2 + 2x - 6$

$= x^2(x - 3) + 2(x - 3)$

$= (x - 3)(x^2 + 2)$

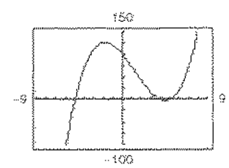
Real zero: 3



76. $x^3 - 4x^2 - 25x + 100 = x^2(x - 4) - 25(x - 4)$

$= (x^2 - 25)(x - 4)$

Real zeros: $\pm 5, 4$



77. $\sum_{i=1}^{6} (2i + 3) = 5 + 7 + 9 + 11 + 13 + 15 = 60$

78. $\sum_{i=0}^{4} 5i^2 = 0 + 5 + 20 + 45 + 80 = 150$

79. $\sum_{k=1}^{10} 15 = 10(15) = 150$

80. $\sum_{k=0}^{8} \frac{3}{k^2 + 1} \approx 5.8791$

Section 11.5 The Area Problem

- You should know the following summation formulas and properties.

  (a) $\sum_{i=1}^{n} c = cn$ (b) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (c) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

  (d) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ (e) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$ (f) $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

- You should be able to evaluate a limit of a summation, $\lim_{n \to \infty} S(n)$.
- You should be able to approximate the area of a region using rectangles. By increasing the number of rectangles, the approximation improves.

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- The area of a plane region above the x-axis bounded by f between $x = a$ and $x = b$ is the limit of the sum of the approximating rectangles:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right)$$

- You should be able to use the limit definition of area to find the area bounded by simple functions in the plane.

Vocabulary Check

1. $\dfrac{n(n+1)}{2}$ 2. $\dfrac{n^2(n+1)^2}{4}$ 3. area

1. $\displaystyle\sum_{i=1}^{60} 7 = 7(60) = 420$

2. $\displaystyle\sum_{i=1}^{45} 3 = 3(45) = 135$

3. $\displaystyle\sum_{i=1}^{20} i^3 = \frac{20^2(21)^2}{4} = 44{,}100$

4. $\displaystyle\sum_{i=1}^{30} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{30(31)(61)}{6} = 9455$

5. $\displaystyle\sum_{k=1}^{20} (k^3+2) = \frac{20^2(21)^2}{4} + 2(20) = 44{,}100 + 40 = 44{,}140$

6. $\displaystyle\sum_{k=1}^{50} (2k+1) = 2\sum_{k=1}^{50} k + \sum_{k=1}^{50} 1 = 2\frac{50(51)}{2} + 50 = 2600$

7. $\displaystyle\sum_{j=1}^{25} (j^2+j) = \frac{25(26)(51)}{6} + \frac{25(26)}{2} = 5850$

8. $\displaystyle\sum_{j=1}^{10} (j^3-3j^2) = \frac{10^2(11)^2}{4} - 3\left(\frac{10(11)(21)}{6}\right) = 1870$

9. (a) $\displaystyle S(n) = \sum_{i=1}^{n} \frac{i^3}{n^4} = \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \frac{n^2+2n+1}{4n^2}$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	1	0.3025	0.255025	0.25050025	0.25005

(c) $\displaystyle\lim_{n\to\infty} S(n) = \frac{1}{4}$

10. (a) $\displaystyle S(n) = \sum_{i=1}^{n} \frac{i}{n^2} = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{n+1}{2n}$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	1	0.55	0.505	0.5005	0.50005

(c) $\displaystyle\lim_{n\to\infty} S(n) = \tfrac{1}{2}$

11. (a) $\displaystyle S(n) = \sum_{i=1}^{n} \frac{3}{n^3}(1+i^2) = \frac{3}{n^3}\left[n + \frac{n(n+1)(2n+1)}{6}\right] = \frac{3}{n^2} + \frac{6n^2+9n+3}{6n^2} = \frac{2n^2+3n+7}{2n^2}$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	6	1.185	1.0154	1.0015	1.00015

(c) $\displaystyle\lim_{n\to\infty} S(n) = 1$

12. (a) $S(n) = \sum_{i=1}^{n} \frac{2i+3}{n^2} = \frac{1}{n^2}\left(2\left(\frac{n(n+1)}{2}\right) + 3n\right) = \frac{n+1}{n} + \frac{3}{n} = \frac{n+4}{n}$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	5	1.4	1.04	1.004	1.0004

(c) $\lim_{n\to\infty} S(n) = 1$

13. (a) $S(n) = \sum_{i=1}^{n}\left(\frac{i^2}{n^3} + \frac{2}{n}\right)\left(\frac{1}{n}\right) = \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6n^3} + \frac{2n}{n}\right] = \frac{1}{6n^3}(2n^2 + 3n + 1) + \frac{2}{n} = \frac{14n^2 + 3n + 1}{6n^3}$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	3	0.2385	0.02338	0.00233	0.0002333

(c) $\lim_{n\to\infty} S(n) = 0$

14. (a) $S(n) = \sum_{i=1}^{n}\left[3 - 2\left(\frac{i}{n}\right)\right]\frac{1}{n} = \frac{1}{n}\left[3n - \frac{2}{n}\frac{n(n+1)}{2}\right] = 3 - \frac{n+1}{n} = \frac{2n-1}{n}$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	1	1.9	1.99	1.999	1.9999

(c) $\lim_{n\to\infty} S(n) = 2$

15. (a) $S(n) = \sum_{i=1}^{n}\left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) = \frac{1}{n}\left[n - \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right] = 1 - \frac{2n^2 + 3n + 1}{6n^2} = \frac{4n^2 - 3n - 1}{6n^2}$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	0	0.615	0.66165	0.66617	0.666617

(c) $\lim_{n\to\infty} S(n) = \frac{2}{3}$

16. (a) $S(n) = \sum_{i=1}^{n}\left(\frac{4}{n} + \frac{2i}{n^2}\right)\left(\frac{2i}{n}\right) = \frac{2}{n}\left[\frac{4}{n}\frac{n(n+1)}{2} + \frac{2}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{2}{n}\left[\frac{4n^2 + 4n}{2n} + \frac{2(2n^2 + 3n + 1)}{6n}\right] = \frac{16n^2 + 18n + 2}{3n^2}$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	12.0	5.94	5.3934	5.3393	5.33393

(c) $\lim_{n\to\infty} S(n) = \frac{16}{3}$

17. $f(x) = x + 4, [-1, 2], n = 6$, width $= \frac{1}{2}$

Area $\approx \frac{1}{2}[3.5 + 4 + 4.5 + 5 + 5.5 + 6] = 14.25$ square units

18. $f(x) = 2 - x^2, -1 \le x \le 1, n = 4$, width $= \frac{1}{2}$

Area $\approx \frac{1}{2}\left[\left(2 - \left(-\frac{1}{2}\right)^2\right) + (2 - 0^2) + \left(2 - \left(\frac{1}{2}\right)^2\right) + (2 - 1^2)\right]$

$= \frac{1}{2}[1.75 + 2 + 1.75 + 1]$

$= 3.25$ square units

19. The width of each rectangle is $\frac{1}{4}$. The height is obtained by evaluating f at the right-hand endpoint of each interval.

$$A \approx \sum_{i=1}^{8} f\left(\frac{i}{4}\right)\left(\frac{1}{4}\right) = \sum_{i=1}^{8} \frac{1}{4}\left(\frac{i}{4}\right)^3\left(\frac{1}{4}\right)$$

$= 1.265625$ square units

20. Area $\approx \frac{1}{2}\left[\frac{1}{16} + \frac{1}{2} + \frac{27}{16} + 4\right]$

$= 3.125$ square units

21. Width of each rectangle is $12/n$. The height is

$$f\left(\frac{12}{n}i\right) = -\frac{1}{3}\left(\frac{12}{n}i\right) + 4.$$

$$A = \sum_{i=1}^{n}\left[-\frac{1}{3}\left(\frac{12i}{n}\right) + 4\right]\left(\frac{12}{n}\right)$$

Note: Exact area is 24.

n	4	8	20	50
Approximate area	18	21	22.8	23.52

22. The width of each rectangle is $3/n$. The height is

$$f\left(\frac{3i}{n}\right) = 9 - \left(\frac{3i}{n}\right)^2.$$

$$A \approx \sum_{i=1}^{n}\left(9 - \left(\frac{3i}{n}\right)^2\right)\frac{3}{n}$$

Note: Exact area is 18.

n	4	8	20	50
Approximate area	14.344	16.242	17.314	17.7282

23. The width of each rectangle is $3/n$. The height is

$$\frac{1}{9}\left(\frac{3i}{n}\right)^3.$$

$$A \approx \sum_{i=1}^{n}\frac{1}{9}\left(\frac{3i}{n}\right)^3\left(\frac{3}{n}\right)$$

n	4	8	20	50
Approximate area	3.52	2.85	2.48	2.34

24. The width of each rectangle is $(2 - (-1))/n = 3/n$. The height is

$$f\left(-1 + \frac{3i}{n}\right) = 3 - \frac{1}{4}\left(-1 + \frac{3i}{n}\right)^3.$$

$$A \approx \sum_{i=1}^{n}\left[3 - \frac{1}{4}\left(-1 + \frac{3i}{n}\right)^3\right]\frac{3}{n}$$

Note: Exact area is $8\frac{1}{16} = 8.0625$.

n	4	8	20	50
Approximate area	7.113	7.614	7.8895	7.994

25. $f(x) = 2x + 5$, $[0, 4]$

The width of each rectangle is $4/n$. The height is

$$f\left(\frac{4i}{n}\right) = 2\left(\frac{4i}{n}\right) + 5 = \frac{8i}{n} + 5.$$

$$A \approx \sum_{i=1}^{n}\left(\frac{8i}{n} + 5\right)\left(\frac{4}{n}\right)$$

n	4	8	20	50	100	∞
Area	40	38	36.8	36.32	36.16	36

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25. —CONTINUED—

$$A \approx \sum_{i=1}^{n}\left(\frac{8i}{n}+5\right)\left(\frac{4}{n}\right) = \sum_{i=1}^{n}\left(\frac{20}{n}+\frac{32}{n^2}i\right) \approx \frac{20}{n}(n) + \frac{32}{n^2}\left(\frac{n(n+1)}{2}\right) = 20 + 16\left(\frac{n^2+n}{n^2}\right)$$

$$A = \lim_{n\to\infty}\left[20 + 16\left(\frac{n^2+n}{n^2}\right)\right] = 20 + 16 = 36$$

26. $f(x) = 3x + 1,\ [0, 4]$

The width of each rectangle is $4/n$. The height is

$$f\left(\frac{4i}{n}\right) = 3\left(\frac{4i}{n}\right) + 1.$$

n	4	8	20	50	100	∞
Area	34	31	29.2	28.48	28.24	28

$$A \approx \sum_{i=1}^{n}\left(\frac{12i}{n}+1\right)\frac{4}{n}$$

$$A \approx \sum_{i=1}^{n}\left(\frac{48i}{n^2}+\frac{4}{n}\right) = \frac{48}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{4}{n}(n) = 24\left(\frac{n^2+n}{n^2}\right) + 4$$

$$A = \lim_{n\to\infty}\left[24\left(\frac{n^2+n}{n^2}\right) + 4\right] = 28$$

27. $f(x) = 16 - 2x,\ [1, 5]$

The width of each rectangle is $4/n$. The height is

$$f\left(1+\frac{4i}{n}\right) = 16 - 2\left(1+\frac{4i}{n}\right) = 14 - \frac{8i}{n}.$$

n	4	8	20	50	100	∞
Area	36	38	39.2	39.68	39.84	40

$$A \approx \sum_{i=1}^{n}\left(14 - \frac{8i}{n}\right)\frac{4}{n}$$

$$A \approx \sum_{i=1}^{n}\left(\frac{56}{n} - \frac{32i}{n^2}\right) = \frac{56}{n}(n) - \frac{32}{n^2}\left(\frac{n(n+1)}{2}\right) = 56 - 16\left(\frac{n(n+1)}{n^2}\right)$$

$$A = \lim_{n\to\infty}\left[56 - 16\left(\frac{n(n+1)}{n^2}\right)\right] = 56 - 16 = 40$$

28. $f(x) = 20 - 2x,\ [2, 6]$

The width of each rectangle is $4/n$. The height is

$$f\left(2+\frac{4i}{n}\right) = 20 - 2\left(2+\frac{4i}{n}\right) = 16 - \frac{8i}{n}.$$

n	4	8	20	50	100	∞
Area	44	46	47.2	47.68	47.84	48

$$A \approx \sum_{i=1}^{n}\left(16 - \frac{8i}{n}\right)\frac{4}{n}$$

$$A \approx \sum_{i=1}^{n}\left(\frac{64}{n} - \frac{32i}{n^2}\right) = \frac{64}{n}(n) - \frac{32}{n^2}\left(\frac{n(n+1)}{2}\right) = 64 - 16\left(\frac{n(n+1)}{n^2}\right)$$

$$A = \lim_{n\to\infty}\left[64 - 16\left(\frac{n(n+1)}{n^2}\right)\right] = 64 - 16 = 48$$

29. $f(x) = 9 - x^2$, $[0, 2]$

The width of each rectangle is $2/n$. The height is

$$f\left(\frac{2i}{n}\right) = 9 - \left(\frac{2i}{n}\right)^2 = 9 - \frac{4i^2}{n^2}.$$

$$A \approx \sum_{i=1}^{n}\left(9 - \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

n	4	8	20	50	100	∞
Area	14.25	14.8125	15.13	15.2528	15.2932	$\frac{46}{3}$

$$A \approx \sum_{i=1}^{n}\left(\frac{18}{n} - \frac{8i^2}{n^3}\right) = \left(\frac{18}{n}\right)n - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = 18 - \frac{4}{3}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$A = \lim_{n\to\infty}\left[18 - \frac{4}{3}\left(\frac{n(n+1)(2n+1)}{n^3}\right)\right] = 18 - \frac{8}{3} = \frac{46}{3}$$

30. $f(x) = x^2 + 1$, $[4, 6]$

The width of each rectangle is $2/n$. The height is

$$f\left(4 + \frac{2i}{n}\right) = \left(4 + \frac{2i}{n}\right)^2 + 1 = 17 + \frac{16i}{n} + \frac{4i^2}{n^2}.$$

$$A \approx \sum_{i=1}^{n}\left(17 + \frac{16i}{n} + \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

n	4	8	20	50	100	∞
Area	57.75	55.1875	53.67	53.0672	52.8668	$\frac{158}{3}$

$$A \approx \sum_{i=1}^{n}\left(\frac{34}{n} + \frac{32i}{n^2} + \frac{8i^2}{n^3}\right) = \left(\frac{34}{n}\right)n + \left(\frac{32}{n^2}\right)\frac{n(n+1)}{2} + \left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6}$$

$$= 34 + 16\left(\frac{n^2+n}{n^2}\right) + \frac{4}{3}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$A = \lim_{n\to\infty}\left[34 + 16\left(\frac{n^2+n}{n^2}\right) + \frac{4}{3}\left(\frac{n(n+1)(2n+1)}{n^3}\right)\right] = 34 + 16 + \frac{8}{3} = \frac{158}{3}$$

31. $f(x) = \frac{1}{2}x + 4$, $[-1, 3]$

The width of each rectangle is $4/n$. The height is

$$f\left(-1 + \frac{4i}{n}\right) = \frac{1}{2}\left(-1 + \frac{4i}{n}\right) + 4 = \frac{7}{2} + \frac{2i}{n}.$$

$$A \approx \sum_{i=1}^{n}\left(\frac{7}{2} + \frac{2i}{n}\right)\left(\frac{4}{n}\right)$$

$$A \approx \sum_{i=1}^{n}\left(\frac{14}{n} + \frac{8i}{n^2}\right) = \left(\frac{14}{n}\right)n + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$A = \lim_{n\to\infty}\left[14 + \frac{4}{n^2}\left(\frac{n(n+1)}{1}\right)\right] = 14 + 4 = 18$$

n	4	8	20	50	100	∞
Area	19	18.5	18.2	18.08	18.04	18

32. $f(x) = \frac{1}{2}x + 1, \ [-2, 2]$

The width of each rectangle is $4/n$. The height is

$$f\left(-2 + \frac{4i}{n}\right) = \frac{1}{2}\left(-2 + \frac{4i}{n}\right) + 1 = \frac{2i}{n}.$$

$$A \approx \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{4}{n}\right) = \sum_{i=1}^{n} \frac{8i}{n^2}$$

$$A \approx \sum_{i=1}^{n} \frac{8i}{n^2} = \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$A = \lim_{n\to\infty} \left[\frac{8}{n^2}\left(\frac{n(n+1)}{2}\right)\right] = 4$$

n	4	8	20	50	100	∞
Area	5	4.5	4.2	4.08	4.04	4

33. $A \approx \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)$

$$= \sum_{i=1}^{n} \left[4\left(\frac{i}{n}\right) + 1\right]\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \left[\frac{4}{n}i + 1\right]$$

$$= \frac{1}{n}\left[\frac{4}{n}\frac{n(n+1)}{2} + n\right]$$

$$= \frac{1}{n}[2(n+1) + n]$$

$$= \frac{3n+2}{n}$$

$$A = \lim_{n\to\infty} \frac{3n+2}{n} = 3 \text{ square units}$$

34. $A \approx \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$$= \sum_{i=1}^{n} \left[3\left(\frac{2i}{n}\right) + 2\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} \left(\frac{6}{n}i + 2\right)$$

$$= \frac{2}{n}\left[\frac{6}{n}\frac{n(n+1)}{2} + 2n\right]$$

$$= 6\left(\frac{n+1}{n}\right) + 4$$

$$A = \lim_{n\to\infty} \left[6\frac{n+1}{n} + 4\right] = 10 \text{ square units}$$

35. $A \approx \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)$

$$= \sum_{i=1}^{n} \left[-2\left(\frac{i}{n}\right) + 3\right]\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \left[-\frac{2i}{n} + 3\right]$$

$$= \frac{1}{n}\left[-\frac{2}{n}\frac{n(n+1)}{2} + 3n\right]$$

$$= \frac{1}{n}[2n - 1]$$

$$A = \lim_{n\to\infty} \frac{2n-1}{n} = 2 \text{ square units}$$

36. $A \approx \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$

$$= \sum_{i=1}^{n} \left[3\left(2 + \frac{3i}{n}\right) - 4\right]\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n} \left[2 + \frac{9}{n}i\right]$$

$$= \frac{3}{n}\left[2n + \frac{9}{n}\frac{n(n+1)}{2}\right]$$

$$= 6 + \frac{27}{2}\frac{(n+1)}{n}$$

$$A = \lim_{n\to\infty} \left[6 + \frac{27}{2}\frac{(n+1)}{n}\right] = \frac{39}{2} \text{ square units}$$

37. $A \approx \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$$= \sum_{i=1}^{n} \left[2 - \left(-1 + \frac{2i}{n}\right)^2\right]\frac{2}{n}$$

$$= \sum_{i=1}^{n}\left[2 - 1 + \frac{4i}{n} - \frac{4i^2}{n^2}\right]\left(\frac{2}{n}\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{2}{n}(n) + \frac{8}{n^2}\frac{n(n+1)}{2} - \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$A = \lim_{n\to\infty}\left[2 + 4\frac{n(n+1)}{n^2} - \frac{4}{3}\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$= 2 + 4 - \frac{8}{3} = \frac{10}{3} \text{ square units}$$

38. $A \approx \sum_{n=1}^{\infty} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)$

$$= \sum_{n=1}^{\infty}\left[\left(\frac{i}{n}\right)^2 + 2\right]\frac{1}{n}$$

$$= \frac{1}{n}\left[\frac{1}{n^2}\frac{n(n+1)(2n+1)}{6} + 2n\right]$$

$$= \frac{(n+1)(2n+1)}{6n^2} + 2$$

$$A = \lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{6n^2} + 2\right] = \frac{7}{3} \text{ square units}$$

39. $A \approx \sum_{i=1}^{n} g\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$

$$= \sum_{i=1}^{n}\left[8 - \left(1 + \frac{i}{n}\right)^3\right]\frac{1}{n}$$

$$= \sum_{i=1}^{n}\left[7 - \frac{3i}{n} - \frac{3i^2}{n^2} - \frac{i^3}{n^3}\right]\frac{1}{n}$$

$$= \frac{7}{n}\sum_{i=1}^{n} 1 - \frac{3}{n^2}\sum_{i=1}^{n} i - \frac{3}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{7}{n}(n) - \frac{3}{n^2}\frac{n(n+1)}{2} - \frac{3}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{1}{n^4}\frac{n^2(n+1)^2}{4}$$

$$A = \lim_{n\to\infty}\left[7 - \frac{3}{2}\frac{n(n+1)}{n^2} - \frac{1}{2n^3}n(n+1)(2n+1) - \frac{1}{n^4}\frac{n^2(n+1)^2}{4}\right] = 7 - \frac{3}{2} - 1 - \frac{1}{4} = \frac{17}{4} \text{ square units}$$

40. $A \approx \sum_{i=1}^{n} g\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$

$$= \sum_{i=1}^{n}\left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[63 - \frac{9i}{n} - \frac{27i^2}{n^2} - \frac{27i^3}{n^3}\right]$$

$$= \frac{3}{n}\left[63n - \frac{9n(n+1)}{2n} - \frac{27n(n+1)(2n+1)}{6n^2} - \frac{27n^2(n+1)^2}{4n^3}\right]$$

$$= 189 - \frac{27(n+1)}{2n} - \frac{27(n+1)(2n+1)}{2n^2} - \frac{81(n+1)^2}{4n^2}$$

$$A = \lim_{n\to\infty}\left[189 - \frac{27(n+1)}{2n} - \frac{27(n+1)(2n+1)}{2n^2} - \frac{81(n+1)^2}{4n^2}\right] = \frac{513}{4} \text{ square units}$$

41. $A \approx \sum_{i=1}^{n} g\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)$

$= \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$

$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{2}{n}i - \frac{1}{n^3}i^3\right]$

$= \frac{1}{n}\left[\frac{2}{n}\frac{n(n+1)}{2} - \frac{1}{n^3}\frac{n^2(n+1)^2}{4}\right]$

$= \frac{n+1}{n} - \frac{(n+1)^2}{4n^2}$

$A = \lim_{n\to\infty}\left[\frac{n+1}{n} - \frac{(n+1)^2}{4n^2}\right]$

$= 1 - \frac{1}{4} = \frac{3}{4}$ square units

42. $A \approx \sum_{i=1}^{n} g\left(2\frac{i}{n}\right)\left(\frac{2}{n}\right)$

$= \sum_{i=1}^{n}\left[4\left(\frac{2i}{n}\right) - \left(\frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$

$= \frac{16}{n^2}\sum_{i=1}^{n} i - \frac{16}{n^4}\sum_{i=1}^{n} i^3$

$= \frac{16}{n^2}\frac{n(n+1)}{2} - \frac{16}{n^4}\frac{n^2(n+1)^2}{4}$

$A = \lim_{n\to\infty}\left[8\frac{n(n+1)}{n^2} - 4\frac{n^2(n+1)^2}{n^4}\right]$

$= 8 - 4 = 4$ square units

43. $A \approx \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$

$= \sum_{i=1}^{n}\left[\frac{1}{4}\left(1 + \frac{3i}{n}\right)^2 + \left(1 + \frac{3i}{n}\right)\right]\left(\frac{3}{n}\right)$

$= \sum_{i=1}^{n}\left(\frac{1}{4} + \frac{3}{2}\frac{i}{n} + \frac{9}{4}\frac{i^2}{n^2} + 1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$

$= \frac{15}{4n}\sum_{i=1}^{n} 1 + \frac{27}{2n^2}\sum_{i=1}^{n} i + \frac{27}{4n^3}\sum_{i=1}^{n} i^2$

$= \frac{15}{4n}(n) + \frac{27}{2n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{4n^3}\frac{n(n+1)(2n+1)}{6}$

$A = \lim_{n\to\infty}\left[\frac{15}{4} + \frac{27}{4}\frac{n(n+1)}{n^2} + \frac{9}{8n^3}n(n+1)(2n+1)\right]$

$= \frac{15}{4} + \frac{27}{4} + \frac{9}{4} = \frac{51}{4}$ square units

44. $A \approx \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$= \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$

$= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\frac{2}{n}$

$= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\frac{2}{n}$

$= \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^4}\sum_{i=1}^{n} i^3$

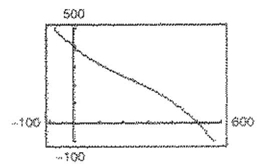
$= \frac{4}{n}(n) - \frac{20}{n^2}\frac{n(n+1)}{2} + \frac{32}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\frac{n^2(n+1)^2}{4}$

$\lim_{n\to\infty} A = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$ square units

45. $y = (-3.0 \cdot 10^{-6})x^3 + 0.002x^2 - 1.05x + 400$

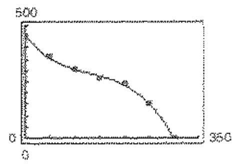
Note that $y = 0$ when $x = 500$.

Area $\approx$ 105,208.33 square feet $\approx$ 2.4153 acres



46. (a) $-4.089 \times 10^{-5}x^3 + 0.01615x^2 - 2.6716x + 452.9286$

(b)



(c) Using a graphing utility to integrate from 0 to 300 gives Area $\approx$ 78,204 square feet.

(Answers will vary.)

47. True. See Formula 2, page 820.

48. False. n approaches infinity.

49. Answers will vary.

50. Area is approximately a triangle of base 2 and height 3.

Area $\approx$ 4; (c)

51. $2\tan x = \tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

$\tan x = 0 \implies x = n\pi$

52.
$$\begin{aligned} \cos 2x - 3\sin x &= 2 \\ 1 - 2\sin^2 x - 3\sin x &= 2 \\ 2\sin^2 x + 3\sin x + 1 &= 0 \\ (2\sin x + 1)(\sin x + 1) &= 0 \end{aligned}$$

$$\sin x = \frac{-1}{2} \implies x = \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi$$

$$\sin x = -1 \implies x = \frac{3\pi}{2} + 2n\pi$$

53.
$$\begin{aligned} (\mathbf{u} \cdot \mathbf{v})\mathbf{u} &= (\langle 4, -5\rangle \cdot \langle -1, -2\rangle)\langle 4, -5\rangle \\ &= 6\langle 4, -5\rangle \\ &= \langle 24, -30\rangle \end{aligned}$$

54.
$$\begin{aligned} 3\mathbf{u} \cdot \mathbf{v} = 3\langle 4, -5\rangle \cdot \langle -1, -2\rangle &= 3(-4 + 10) \\ &= 18 \end{aligned}$$

55. $\|\mathbf{v}\| - 2 = \sqrt{5} - 2$

56.
$$\begin{aligned} \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 &= (4^2 + (-5)^2) - ((-1)^2 + (-2)^2) \\ &= (16 + 25) - (1 + 4) \\ &= 36 \end{aligned}$$

Review Exercises for Chapter 11

1. $\lim_{x\to 3} (6x - 1)$

The limit (17) can be reached.

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	16.4	16.94	16.994	17	17.006	17.06	17.6

2. $f(x) = \dfrac{x - 2}{3x^2 - 4x - 4}$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.1299	0.1255	0.1250	Undef.	0.1250	0.1245	0.1205

$\lim_{x\to 2} f(x) = \dfrac{1}{8}$

The limit cannot be reached.

3. $f(x) = \dfrac{1 - e^{-x}}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	1.0517	1.0050	1.0005	Error	0.9995	0.9950	0.9516

$\lim_{x\to 0} \dfrac{1 - e^{-x}}{x} = 1$

The limit cannot be reached.

4. $f(x) = \dfrac{\ln(1 - x)}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.9531	-0.9950	-0.9995	Error	-1.0005	-1.0050	-1.0536

$\lim_{x\to 0} \dfrac{\ln(1 - x)}{x} = -1$

The limit cannot be reached.

5. $\lim_{x\to 1} (3 - x) = 2$

6. Limit does not exist.

7. $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1} = 2$

8. $\lim_{x\to -1} (2x^2 + 1) = 3$

9. (a) $\lim_{x\to c} [f(x)]^3 = 4^3 = 64$

(b) $\lim_{x\to c} [3f(x) - g(x)] = 3(4) - 5 = 7$

(c) $\lim_{x\to c} [f(x)g(x)] = (4)(5) = 20$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{4}{5}$

10. (a) $\lim_{x\to c} \sqrt[3]{f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c} \dfrac{f(x)}{18} = \dfrac{27}{18} = \dfrac{3}{2}$

(c) $\lim_{x\to c} [f(x)\,g(x)] = (27)(12) = 324$

(d) $\lim_{x\to c} [f(x) - 2g(x)] = 27 - 2(12) = 3$

11. $\lim_{x\to 4} \left(\frac{1}{2}x + 3\right) = \frac{1}{2}(4) + 3 = 5$

12. $\lim_{x\to 3} (5x - 4) = 5(3) - 4 = 11$

13. $\lim_{x\to 2} (5x - 3)(3x + 5) = (5(2) - 3)(3(2) + 5)$
$= (7)(11) = 77$

14. $\lim_{x\to -2} (5 - 2x - x^2) = 5 - 2(-2) - (-2)^2 = 5$

15. $\lim_{t\to 3} \dfrac{t^2 + 1}{t} = \dfrac{9 + 1}{3} = \dfrac{10}{3}$

16. $\lim_{x\to 2} \dfrac{3x + 5}{5x - 3} = \dfrac{3(2) + 5}{5(2) - 3} = \dfrac{11}{7}$

17. $\lim_{x \to -2} \sqrt[3]{4x} = (-8)^{1/3} = -2$

18. $\lim_{x \to -1} \sqrt{5-x} = \sqrt{5-(-1)} = \sqrt{6}$

19. $\lim_{x \to \pi} \sin 3x = \sin 3\pi = 0$

20. $\lim_{x \to 0} \tan x = \tan 0 = 0$

21. $\lim_{x \to -1} 2e^x = 2e^{-1} = \dfrac{2}{e}$

22. $\lim_{x \to 4} \ln x = \ln 4$

23. $\lim_{x \to -1/2} \arcsin x = \arcsin\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$

24. $\lim_{x \to 0} \arctan x = \arctan 0 = 0$

25. $$\lim_{t \to -2} \frac{t+2}{t^2-4} = \lim_{t \to -2} \frac{t+2}{(t+2)(t-2)} = \lim_{t \to -2} \frac{1}{t-2} = -\frac{1}{4}$$

26. $$\lim_{t \to 3} \frac{t^2-9}{t-3} = \lim_{t \to 3} \frac{(t-3)(t+3)}{t-3} = \lim_{t \to 3} (t+3) = 6$$

27. $$\lim_{x \to 5} \frac{x-5}{x^2+5x-50} = \lim_{x \to 5} \frac{x-5}{(x-5)(x+10)} = \lim_{x \to 5} \frac{1}{x+10} = \frac{1}{15}$$

28. $$\lim_{x \to -1} \frac{x+1}{(x^2-5x-6)} = \lim_{x \to -1} \frac{(x+1)}{(x+1)(x-6)} = \lim_{x \to -1} \frac{1}{x-6} = -\frac{1}{7}$$

29. $$\lim_{x \to -2} \frac{x^2-4}{x^3+8} = \lim_{x \to -2} \frac{(x+2)(x-2)}{(x+2)(x^2-2x+4)} = \lim_{x \to -2} \frac{x-2}{x^2-2x+4} = \frac{-4}{12} = \frac{-1}{3}$$

30. $$\lim_{x \to 4} \frac{x^3-64}{x^2-16} = \lim_{x \to 4} \frac{(x-4)(x^2+4x+16)}{(x-4)(x+4)} = \lim_{x \to 4} \frac{x^2+4x+16}{x+4} = \frac{16+16+16}{8} = 6$$

31. $$\lim_{x \to -1} \frac{1/(x+2)-1}{x+1} = \lim_{x \to -1} \frac{1-(x+2)}{(x+2)(x+1)} = \lim_{x \to -1} \frac{-(x+1)}{(x+2)(x+1)} = \lim_{x \to -1} \frac{-1}{(x+2)} = -1$$

32. $$\lim_{x \to 0} \frac{(1/(1+x)-1)}{x} = \lim_{x \to 0} \frac{1-(1+x)}{x(1+x)} = \lim_{x \to 0} \frac{-x}{x(1+x)} = -1$$

33. $$\lim_{u \to 0} \frac{\sqrt{4+u}-2}{u} = \lim_{u \to 0} \frac{\sqrt{4+u}-2}{u} \cdot \frac{\sqrt{4+u}+2}{\sqrt{4+u}+2} = \lim_{u \to 0} \frac{(4+u)-4}{u\left(\sqrt{4+u}+2\right)} = \lim_{u \to 0} \frac{1}{\sqrt{4+u}+2} = \frac{1}{4}$$

34. $$\lim_{v \to 0} \frac{\sqrt{v+9}-3}{v} = \lim_{v \to 0} \frac{\sqrt{v+9}-3}{v} \cdot \frac{\sqrt{v+9}+3}{\sqrt{v+9}+3} = \lim_{v \to 0} \frac{(v+9)-9}{v\left[\sqrt{v+9}+3\right]} = \lim_{v \to 0} \frac{1}{\sqrt{v+9}+3} = \frac{1}{6}$$

35. $\lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5} = \lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5} \cdot \dfrac{\sqrt{x-1}+2}{\sqrt{x-1}+2}$

$= \lim\limits_{x\to 5} \dfrac{(x-1)-4}{(x-5)\left(\sqrt{x-1}+2\right)}$

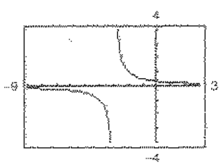
$= \lim\limits_{x\to 5} \dfrac{1}{\sqrt{x-1}+2} = \dfrac{1}{2+2} = \dfrac{1}{4}$

36. $\lim\limits_{x\to 1} \dfrac{\sqrt{3}-\sqrt{x+2}}{1-x} = \lim\limits_{x\to 1} \dfrac{\sqrt{3}-\sqrt{x+2}}{1-x} \cdot \dfrac{\sqrt{3}+\sqrt{x+2}}{\sqrt{3}+\sqrt{x+2}}$

$= \lim\limits_{x\to 1} \dfrac{3-(x+2)}{(1-x)\left(\sqrt{3}+\sqrt{x+2}\right)}$

$= \lim\limits_{x\to 1} \dfrac{1}{\sqrt{3}+\sqrt{x+2}} = \dfrac{1}{2\sqrt{3}} = \dfrac{\sqrt{3}}{6}$

37. (a)

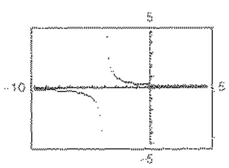


$\lim\limits_{x\to 3} \dfrac{x-3}{x^2-9} = \dfrac{1}{6}$

(b)

x	2.9	2.99	3	3.01	3.1
$f(x)$	0.1695	0.1669	Error	0.1664	0.1639

38. (a)

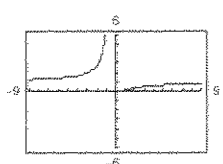


$\lim\limits_{x\to 4} \dfrac{4-x}{16-x^2} = \dfrac{1}{8}$

(b)

x	3.99	3.999	4	4.001	4.01
y_1	0.12516	0.12502	Error	0.12498	0.12484

39. (a)

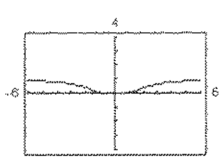


$\lim\limits_{x\to 0} e^{-2/x}$ does not exist.

(b) Answers will vary.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
y_1	4.85 E 8	7.2 E 86	Error	Error	0	1 E -87	2.1 E -9

40. (a)

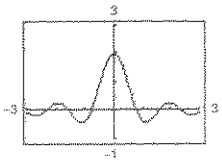


$\lim\limits_{x\to 0} e^{-4/x^2} = 0$

(b)

x	-0.01	-0.001	0	0.001	0.01
y_1	0	0	Error	0	0

41. (a)

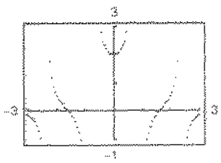


$$\lim_{x\to 0}\frac{\sin 4x}{2x} = 2$$

(b)

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
y_1	1.9471	1.9995	1.999995	error	1.999995	1.995	1.9471

42. (a)

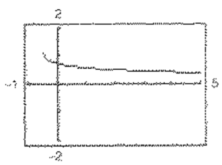


$$\lim_{x\to 0}\frac{\tan 2x}{x} = 2$$

(b)

x	-0.01	-0.001	0	0.001	0.01
y_1	2.0003	2	Error	2	2.0003

43. (a)



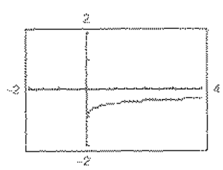
$$\lim_{x\to 1^-}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1} \approx 0.577$$

$$\left(\text{Exact value: } \frac{\sqrt{3}}{3}\right)$$

(b)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

44. (a)



$$\lim_{x\to 1^+}\frac{1-\sqrt{x}}{x-1} = -\frac{1}{2}$$

(b)

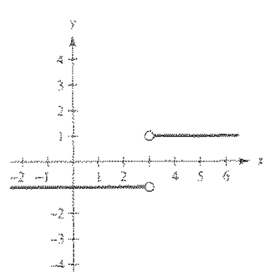
x	1.1	1.01	1.001	1.0001
$f(x)$	-0.4881	-0.4988	-0.4999	-0.5000

45. $f(x) = \dfrac{|x-3|}{x-3}$

Limit does not exist because

$\lim_{x\to 3^+} f(x) = 1$ and

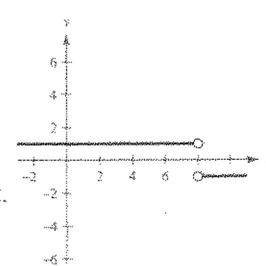
$\lim_{x\to 3^-} f(x) = -1.$



46. $\lim_{x\to 8^-}\dfrac{|8-x|}{8-x} = 1$

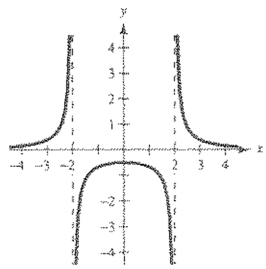
$\lim_{x\to 8^+}\dfrac{|8-x|}{8-x} = -1$

$\lim_{x\to 8}\dfrac{|8-x|}{8-x}$ does not exist.

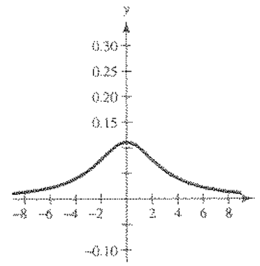


47. $f(x) = \dfrac{2}{x^2 - 4}$

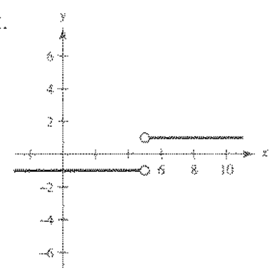
Limit does not exist.



48. $\displaystyle\lim_{x \to -3} \frac{1}{x^2 + 9} = \frac{1}{(-3)^2 + 9} = \frac{1}{18}$



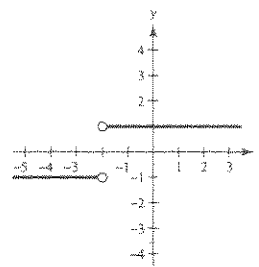
49. $\displaystyle\lim_{x \to 5} \frac{|x - 5|}{x - 5}$ does not exist.



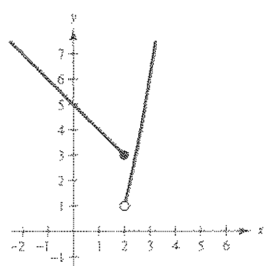
50. $\displaystyle\lim_{x \to -2^-} \frac{|x + 2|}{x + 2} = -1$

$\displaystyle\lim_{x \to -2^+} \frac{|x + 2|}{x + 2} = 1$

$\displaystyle\lim_{x \to -2} \frac{|x + 2|}{x + 2}$ does not exist.



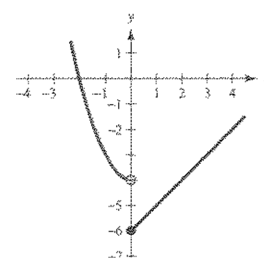
51. $\displaystyle\lim_{x \to 2} f(x)$ does not exist.



52. $\displaystyle\lim_{x \to 0^-} f(x) = -4$

$\displaystyle\lim_{x \to 0^+} f(x) = -6$

$\displaystyle\lim_{x \to 0} f(x)$ does not exist.



53. $$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x + h) - (x + h)^2 - (3x - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{3x + 3h - x^2 - 2xh - h^2 - 3x + x^2}{h} = \lim_{h \to 0} \frac{3h - 2xh - h^2}{h}$$

$$= \lim_{h \to 0} (3 - 2x - h) = 3 - 2x$$

54. $$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x + h)^2 - 5(x + h) - 2] - [x^2 - 5x - 2]}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 5h}{h}$$

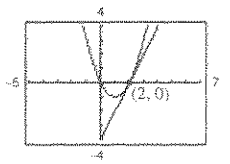
$$= \lim_{h \to 0} [2x + h - 5] = 2x - 5$$

55. Slope ≈ 2

Answers will vary.

56. Slope $= 0$

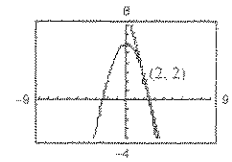
57.



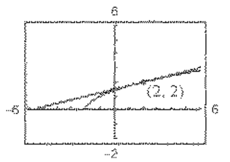
Slope at $(2, f(2))$ is approximately 2.

58. At $(2, f(2)) = (2, 2)$.

Slope $= -4$



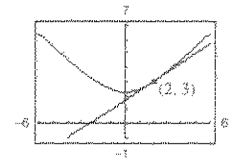
59.



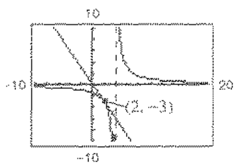
Slope is $\frac{1}{4}$ at $(2, 2)$.

60. At $(2, f(2)) = (2, 3)$.

Slope $= \frac{2}{3}$

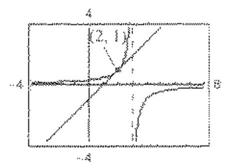


61.



At $(2, f(2)) = (2, -3)$, the slope is approximately -1.5.

62.



At $(2, f(2)) = (2, 1)$, the slope is approximately 1.

63. $$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 - 4x}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h}$$

$$= \lim_{h \to 0} (2x + h - 4) = 2x - 4$$

(a) At $(0, 0)$, $m = 2(0) - 4 = -4$.

(b) At $(5, 5)$, $m = 2(5) - 4 = 6$.

64. $$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(1/4)(x+h)^4 - (1/4)x^4}{h}$$

$$= \lim_{h \to 0} \frac{(1/4)[x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4]}{h}$$

$$= \lim_{h \to 0} \frac{1}{4}[4x^3 + 6x^2h + 4xh^2 + h^3] = x^3$$

(a) At $(-2, 4)$, $m = (-2)^3 = -8$.

(b) At $\left(1, \frac{1}{4}\right)$, $m = (1)^3 = 1$.

65. $$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{4}{x+h-6} - \dfrac{4}{x-6}}{h}$$

$$= \lim_{h \to 0} \frac{4(x-6) - 4(x+h-6)}{(x+h-6)(x-6)h}$$

$$= \lim_{h \to 0} \frac{-4h}{(x+h-6)(x-6)h}$$

$$= \lim_{h \to 0} \frac{-4}{(x+h-6)(x-6)} = \frac{-4}{(x-6)^2}$$

(a) At $(7, 4)$, $m = \dfrac{-4}{(7-6)^2} = -4$.

(b) At $(8, 2)$, $m = \dfrac{-4}{(8-6)^2} = -1$.

66. $m = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \dfrac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}$

$= \lim_{h \to 0} \dfrac{(x + h) - x}{h\left[\sqrt{x + h} + \sqrt{x}\right]}$

$= \lim_{h \to 0} \dfrac{1}{\sqrt{x + h} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

(a) At $(1, 1)$, $m = \dfrac{1}{2\sqrt{1}} = \dfrac{1}{2}$.

(b) At $(4, 2)$, $m = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$.

67. $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \dfrac{5 - 5}{h} = 0$

68. $g'(x) = \lim_{h \to 0} \dfrac{g(x + h) - g(x)}{h}$

$= \lim_{h \to 0} \dfrac{-3 - (-3)}{h} = \lim_{h \to 0} \dfrac{0}{h} = 0$

69. $h'(x) = \lim_{k \to 0} \dfrac{h(x + k) - h(x)}{k}$

$= \lim_{k \to 0} \dfrac{\left[5 - \frac{1}{2}(x + k)\right] - \left[5 - \frac{1}{2}x\right]}{k}$

$= \lim_{k \to 0} \dfrac{-\frac{1}{2}k}{k} = -\dfrac{1}{2}$

70. $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{3(x + h) - 3x}{h} = 3$

71. $g'(x) = \lim_{h \to 0} \dfrac{g(x + h) - g(x)}{h}$

$= \lim_{h \to 0} \dfrac{2(x + h)^2 - 1 - (2x^2 - 1)}{h}$

$= \lim_{h \to 0} \dfrac{2x^2 + 4xh + 2h^2 - 2x^2}{h}$

$= \lim_{h \to 0} (4x + 2h)$

$= 4x$

72. $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \dfrac{-(x + h)^3 + 4(x + h) - (-x^3 + 4x)}{h}$

$= \lim_{h \to 0} \dfrac{-x^3 - 3x^2h - 3xh^2 - h^3 + 4x + 4h + x^3 - 4x}{h}$

$= \lim_{h \to 0} \dfrac{-3x^2h - 3xh^2 - h^3 + 4h}{h}$

$= \lim_{h \to 0} (-3x^2 - 3xh - h^2 + 4)$

$= -3x^2 + 4$

73. $f'(t) = \lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$

$= \lim_{h \to 0} \dfrac{\sqrt{t+h+5} - \sqrt{t+5}}{h} \cdot \dfrac{\sqrt{t+h+5} + \sqrt{t+5}}{\sqrt{t+h+5} + \sqrt{t+5}}$

$= \lim_{h \to 0} \dfrac{(t+h+5) - (t+5)}{h\left(\sqrt{t+h+5} + \sqrt{t+5}\right)}$

$= \lim_{h \to 0} \dfrac{1}{\sqrt{t+h+5} + \sqrt{t+5}}$

$= \dfrac{1}{2\sqrt{t+5}}$

74. $g'(t) = \lim_{h \to 0} \dfrac{g(t+h) - g(t)}{h}$

$= \lim_{h \to 0} \dfrac{\sqrt{t+h-3} - \sqrt{t-3}}{h} \cdot \dfrac{\sqrt{t+h-3} + \sqrt{t-3}}{\sqrt{t+h-3} + \sqrt{t-3}}$

$= \lim_{h \to 0} \dfrac{(t+h-3) - (t-3)}{h\left(\sqrt{t+h-3} + \sqrt{t-3}\right)}$

$= \lim_{h \to 0} \dfrac{1}{\sqrt{t+h-3} + \sqrt{t-3}}$

$= \dfrac{1}{2\sqrt{t-3}}$

75. $g'(s) = \dfrac{g(s+h) - g(s)}{h}$

$= \lim_{h \to 0} \dfrac{\dfrac{4}{s+h+5} - \dfrac{4}{s+5}}{h}$

$= \lim_{h \to 0} \dfrac{4s + 20 - 4s - 4h - 20}{(s+h+5)(s+5)h}$

$= \lim_{h \to 0} \dfrac{-4h}{(s+h+5)(s+5)h}$

$= \lim_{h \to 0} \dfrac{-4}{(s+h+5)(s+5)}$

$= \dfrac{-4}{(s+5)^2}$

76. $g'(t) = \lim_{h \to 0} \dfrac{g(t+h) - g(t)}{h}$

$= \lim_{h \to 0} \dfrac{\dfrac{6}{5-(t+h)} - \dfrac{6}{5-t}}{h}$

$= \lim_{h \to 0} \dfrac{30 - 6t - 30 + 6t + 6h}{h(5-t-h)(5-t)}$

$= \lim_{h \to 0} \dfrac{6}{(5-t-h)(5-t)}$

$= \dfrac{6}{(5-t)^2}$

77. $g'(x) = \lim_{h\to 0} \dfrac{g(x+h) - g(x)}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{x+h+4}} - \dfrac{1}{\sqrt{x+4}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+4} - \sqrt{x+h+4}}{h\sqrt{x+h+4}\sqrt{x+4}} \cdot \frac{\sqrt{x+4} + \sqrt{x+h+4}}{\sqrt{x+4} + \sqrt{x+h+4}}$$

$$= \lim_{h\to 0} \frac{(x+4) - (x+h+4)}{h\sqrt{x+h+4}\sqrt{x+4}\left[\sqrt{x+4} + \sqrt{x+h+4}\right]}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h+4}\sqrt{x+4}\left[\sqrt{x+4} + \sqrt{x+h+4}\right]}$$

$$= \frac{-1}{(x+4)2\sqrt{x+4}}$$

$$= \frac{-1}{2(x+4)^{3/2}}$$

78. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{\sqrt{12-x-h}} - \dfrac{1}{\sqrt{12-x}}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{12-x} - \sqrt{12-x-h}}{h\left[\sqrt{12-x-h}\sqrt{12-x}\right]} \cdot \frac{\sqrt{12-x} + \sqrt{12-x-h}}{\sqrt{12-x} + \sqrt{12-x-h}}$$

$$= \lim_{h\to 0} \frac{(12-x) - (12-x-h)}{h\left[\sqrt{12-x-h}\sqrt{12-x}\right]} \cdot \frac{1}{\left[\sqrt{12-x} + \sqrt{12-x-h}\right]}$$

$$= \lim_{h\to 0} \frac{1}{\left[\sqrt{12-x-h}\sqrt{12-x}\right]\left[\sqrt{12-x} + \sqrt{12-x-h}\right]}$$

$$= \frac{1}{(12-x)2\sqrt{12-x}}$$

$$= \frac{1}{2(12-x)^{3/2}}$$

79. $\lim_{x\to\infty} \dfrac{4x}{2x-3} = \dfrac{4}{2} = 2$

80. $\lim_{x\to\infty} \dfrac{7x}{14x+2} = \dfrac{7}{14} = \dfrac{1}{2}$

81. $\lim_{x\to-\infty} \dfrac{2x}{x^2-25} = 0$

82. $\lim_{x\to-\infty} \dfrac{3x}{(1-x)^3} = 0$

83. $\lim_{x\to\infty} \dfrac{x^2}{2x+3}$ does not exist.

84. $\lim_{y\to\infty} \dfrac{3y^4}{y^2+1}$ does not exist.

85. $\lim_{x\to\infty} \left[\dfrac{x}{(x-2)^2} + 3\right] = 0 + 3 = 3$

86. $\lim_{x\to\infty} \left[2 - \dfrac{2x^2}{(x+1)^2}\right] = 2 - 2 = 0$

87. $a_n = \dfrac{2n - 3}{5n + 4}$

$a_1 = -\dfrac{1}{9}$ $\quad a_4 = \dfrac{5}{24}$

$a_2 = \dfrac{1}{14}$ $\quad a_5 = \dfrac{7}{29}$

$a_3 = \dfrac{3}{19}$

$\lim\limits_{n\to\infty} a_n = \dfrac{2}{5}$

88. $a_n = \dfrac{2n}{n^2 + 1}$

$a_1 = \dfrac{2}{2} = 1$ $\quad a_4 = \dfrac{8}{17}$

$a_2 = \dfrac{4}{5}$ $\quad a_5 = \dfrac{10}{26} = \dfrac{5}{13}$

$a_3 = \dfrac{6}{10} = \dfrac{3}{5}$

$\lim\limits_{n\to\infty} a_n = 0$

89. $a_n = \dfrac{(-1)^n}{n^3}$

$a_1 = -1$ $\quad a_4 = \dfrac{1}{64}$

$a_2 = \dfrac{1}{8}$ $\quad a_5 = -\dfrac{1}{125}$

$a_3 = -\dfrac{1}{27}$

$\lim\limits_{n\to\infty} a_n = 0$

90. $a_n = \dfrac{(-1)^{n+1}}{n}$

$a_1 = 1$ $\quad a_4 = -\dfrac{1}{4}$

$a_2 = -\dfrac{1}{2}$ $\quad a_5 = \dfrac{1}{5}$

$a_3 = \dfrac{1}{3}$

$\lim\limits_{n\to\infty} a_n = 0$

91. $a_n = \dfrac{1}{2n^2}[3 - 2n(n + 1)] = \dfrac{3}{2n^2} - \dfrac{n + 1}{n}$

$-0.5, -1.125, -1.16\overline{6}, -1.15625, -1.14$

$\lim\limits_{n\to\infty} a_n = 0 - 1 = -1$

92. $a_n = 2 + \dfrac{2}{n}(n - 1) - \dfrac{4}{n} = 4 - \dfrac{6}{n}$

$-2, 1, 2, \dfrac{5}{2}, \dfrac{14}{5}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(4 - \dfrac{6}{n}\right) = 4$

93. (a) $\displaystyle\sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{i}{n}\right)\frac{1}{n} = \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^2}\sum_{i=1}^{n} i$

$$= \frac{4}{n^3}\frac{n(n + 1)(2n + 1)}{6} - \frac{1}{n^2}\frac{n(n + 1)}{2}$$

$$= \frac{4n(n + 1)(2n + 1) - 3n^2(n + 1)}{6n^3}$$

$$= \frac{n(n + 1)(8n + 4 - 3n)}{6n^3}$$

$$= \frac{(n + 1)(5n + 4)}{6n^2}$$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	3	0.99	0.8484	0.8348	0.8335

(c) $\lim\limits_{n\to\infty} S(n) = \dfrac{5}{6}$

94. (a) $\sum_{i=1}^{n}\left[4-\left(\frac{3i}{n}\right)^2\right]\left(\frac{3i}{n^2}\right) = \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^4}\sum_{i=1}^{n} i^3 = \frac{12}{n^2}\frac{n(n+1)}{2} - \frac{27}{n^4}\frac{n^2(n+1)^2}{4}$

$$= \frac{24n^2+24n-27(n^2+2n+1)}{4n^2}$$

$$= \frac{-3n^2-30n-27}{4n^2} = \frac{-3}{4n^2}(n^2+10n+9)$$

$$= \frac{-3(n+1)(n+9)}{4n^2}$$

(b)

n	10^0	10^1	10^2	10^3	10^4
$S(n)$	-15	-1.5675	-0.8257	-0.7575	-0.7508

(c) $\lim_{n\to\infty} Sn = -\frac{3}{4}$

95. Area $\approx \frac{1}{2}\left(\frac{7}{2}+3+\frac{5}{2}+2+\frac{3}{2}+1\right)$

$= \frac{1}{2}\left(\frac{27}{2}\right) = \frac{27}{4} = 6.75$

96. Width of rectangle: $\frac{1}{4}$

Height is f evaluated at right endpoint

Area $\approx \frac{1}{4}\left[f\left(\frac{1}{4}\right)+f\left(\frac{1}{2}\right)+f\left(\frac{3}{4}\right)+f(1)\right]$

$= \frac{1}{4}\left[4-\left(\frac{1}{4}\right)^2+4-\left(\frac{1}{2}\right)^2+4-\left(\frac{3}{4}\right)^2+4-1\right]$

$= \frac{1}{4}\left[15-\frac{14}{16}\right] = \frac{113}{32} = 3.53125$

97. $f(x) = \frac{1}{4}x^2,\ b-a = 4-0 = 4$

$$A \approx \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\left(\frac{4}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{4}\left(\frac{4i}{n}\right)^2\left(\frac{4}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{16}{n^2}i^2$$

$$= \frac{16}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{8(n+1)(2n+1)}{3n^2}$$

n	4	8	20	50
Approximate area	7.5	6.375	5.74	5.4944

$\left(\text{Exact area is } \frac{16}{3} \approx 5.33.\right)$

98. $f(x) = 4x - x^2$

n	4	8	20	50
Approximate area	10	10.5	10.64	10.6624

$\left(\text{Exact area is } 10\frac{2}{3}.\right)$

99. $A = \lim_{n\to\infty}\sum_{i=1}^{n}\left(10-\frac{10i}{n}\right)\left(\frac{10}{n}\right)$

$$= \lim_{n\to\infty}\left[\frac{100}{n}\sum_{i=1}^{n}1 - \frac{100}{n^2}\sum_{i=1}^{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{100}{n}(n) - \frac{100}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty}\left[100 - 50\frac{n(n+1)}{n^2}\right]$$

$= 100 - 50 = 50$, exact area

100. $A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[2\left(3 + \frac{3i}{n}\right) - 6\right]\left(\frac{3}{n}\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{18i}{n^2} = \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^{n} i$

$= \lim_{n\to\infty} \frac{18}{n^2} \frac{n(n+1)}{2} = 9$, exact area

101. $A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-1 + \frac{3i}{n}\right)^2 + 4\right]\left(\frac{3}{n}\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[5 - \frac{6i}{n} + \frac{9i^2}{n^2}\right]\frac{3}{n}$

$= \lim_{n\to\infty}\left[\frac{15}{n} \sum_{i=1}^{n} 1 - \frac{18}{n^2} \sum_{i=1}^{n} i + \frac{27}{n^3} \sum_{i=1}^{n} i^2\right]$

$= \lim_{n\to\infty}\left[\frac{15}{n}(n) - \frac{18}{n^2}\frac{n(n+1)}{2} + \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$

$= 15 - 9 + 9 = 15$, exact area

102. $A = \lim_{n\to\infty} \sum_{i=1}^{n} 8\left(\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^2\right)\frac{1}{n}$

$= \lim_{n\to\infty} \left[\frac{8}{n^2} \sum_{i=1}^{n} i - \frac{8}{n^3} \sum i^2\right]$

$= \lim_{n\to\infty} \left[\frac{8}{n^2}\frac{n(n+1)}{2} - \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$

$= 4 - \frac{8}{3} = \frac{4}{3}$, exact area

103. $f(x) = x^3 + 1$, $[0, 2]$

The width of each rectangle is $2/n$. The height is

$f\left(\frac{2i}{n}\right) = \left(\frac{2i}{n}\right)^3 + 1.$

$A \approx \sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3 + 1\right]\left(\frac{2}{n}\right)$

$= \sum_{i=1}^{n}\left[\frac{16i^3}{n^4} + \frac{2}{n}\right]$

$= \left(\frac{16}{n^4}\right)\frac{n^2(n+1)^2}{4} + \left(\frac{2}{n}\right)n$

$= \frac{4}{n^2}(n+1)^2 + 2$

$A = \lim_{n\to\infty} \left[\frac{4}{n^2}(n+1)^2 + 2\right] = 4 + 2 = 6$

104. $f(x) = 1 - x^3$, $[-3, -1]$

The width of each rectangle is $2/n$. The height is

$f\left(-3 + \frac{2i}{n}\right) = 1 - \left(-3 + \frac{2i}{n}\right)^3$

$= 28 - 54\frac{i}{n} + \frac{36i^2}{n^2} - \frac{8i^3}{n^3}.$

$A \approx \sum_{i=1}^{n}\left[28 - \frac{54}{n}i + \frac{36}{n^2}i^2 - \frac{8}{n^3}i^3\right]\left(\frac{2}{n}\right)$

$= \frac{56}{n}(n) - \frac{108}{n^2}\frac{n(n+1)}{2} + \frac{72}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\frac{n^2(n+1)^2}{4}$

$A = \lim_{n\to\infty}\left[56 - 54\frac{n(n+1)}{n^2} + 12\frac{n(n+1)(2n+1)}{n^3} - \frac{4n^2(n+1)^2}{n^4}\right]$

$= 56 - 54 + 24 - 4 = 22$

105. $f(x) = 3(x^3 - x^2),\ [1, 3]$

The width of each rectangle is $2/n$. The height is

$$f\left(1 + \frac{2i}{n}\right) = 3\left[\left(1 + \frac{2i}{n}\right)^3 - \left(1 + \frac{2i}{n}\right)^2\right]$$

$$= \frac{6i}{n} + \frac{24i^2}{n^2} + \frac{24i^3}{n^3}.$$

$$A \approx \sum_{i=1}^{n}\left[\frac{6i}{n} + \frac{24i^2}{n^2} + \frac{24i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2}\frac{n(n+1)}{2} + \frac{48}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{48}{n^4}\frac{n^2(n+1)^2}{4}$$

$$A = \lim_{n\to\infty}\left[6 + 8\frac{n(n+1)(2n+1)}{n^3} + 12\frac{n^2(n+1)^2}{n^4}\right]$$

$$= 6 + 16 + 12 = 34$$

106. $f(x) = 5 - (x + 2)^2,\ [-2, 0]$

The width of each rectangle is $2/n$. The height is

$$f\left(-2 + \frac{2i}{n}\right) = 5 - \left[\left(-2 + \frac{2i}{n}\right) + 2\right]^2$$

$$= 5 - \left(\frac{2i}{n}\right)^2$$

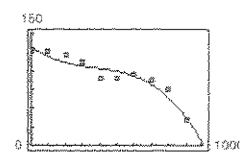
$$= 5 - \frac{4i^2}{n^2}.$$

$$A \approx \sum_{i=1}^{n}\left[5 - \frac{4i^2}{n^2}\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\frac{10}{n} - \frac{8}{n^3}i^2\right]$$

$$A = \lim_{n\to\infty}\left[\frac{10}{n}(n) - \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}\right] = 10 - \frac{8}{3} = \frac{22}{3}$$

107. (a) $y = -3.376 \times 10^{-7}x^3 + 3.753 \times 10^{-4}x^2 - 0.168x + 132.168$

(b)



(c) Area $\approx$ 88,868 square feet; answers will vary.

108. True (assuming all the limits exist)

109. False. The limit does not exist.

110. Answers will vary.

Chapter 11 Practice Test

1. Use a graphing utility to complete the table and use the result to estimate the limit

$$\lim_{x \to 3} \frac{x-3}{x^2-9}.$$

x	2.9	2.99	3	3.01	3.1
$f(x)$			?		

2. Graph the function

$$f(x) = \frac{\sqrt{x+4}-2}{x}$$

and estimate the limit

$$\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}.$$

3. Find the limit $\lim_{x \to 2} e^{x-2}$ by direct substitution.

4. Find the limit $\lim_{x \to 1} \frac{x^3-1}{x-1}$ analytically.

5. Use a graphing utility to estimate the limit

$$\lim_{x \to 0} \frac{\sin 5x}{2x}.$$

6. Find the limit

$$\lim_{x \to -2} \frac{|x+2|}{x+2}.$$

7. Use the limit process to find the slope of the graph of $f(x) = \sqrt{x}$ at the point $(4, 2)$.

8. Find the derivative of the function $f(x) = 3x - 1$.

9. Find the limits.

(a) $\lim_{x \to \infty} \frac{3}{x^4}$ (b) $\lim_{x \to -\infty} \frac{x^2}{x^2+3}$ (c) $\lim_{x \to \infty} \frac{|x|}{1-x}$

10. Write the first four terms of the sequence $a_n = \frac{1-n^2}{2n^2+1}$ and find the limit of the sequence.

11. Find the sum $\sum_{i=1}^{25} (i^2 + i)$.

12. Write the sum $\sum_{i=1}^{n} \frac{i^2}{n^3}$ as a rational function $S(n)$, and find $\lim_{n \to \infty} S(n)$.

13. Find the area of the region bounded by $f(x) = 1 - x^2$ over the interval $0 \le x \le 1$.

APPENDICES